Chapter 03.05
Secant Method of Solving Nonlinear Equations

After reading this chapter, you should be able to:

1. *derive the secant method to solve for the roots of a nonlinear equation,*
2. *use the secant method to numerically solve a nonlinear equation.*

What is the secant method and why would I want to use it instead of the Newton-Raphson method?

The Newton-Raphson method of solving a nonlinear equation $f(x)=0$ is given by the iterative formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1}$$

One of the drawbacks of the Newton-Raphson method is that you have to evaluate the derivative of the function. With availability of symbolic manipulators such as Maple, MathCAD, MATHEMATICA and MATLAB, this process has become more convenient. However, it still can be a laborious process, and even intractable if the function is derived as part of a numerical scheme. To overcome these drawbacks, the derivative of the function, $f(x)$ is approximated as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \tag{2}$$

Substituting Equation (2) in Equation (1) gives

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \tag{3}$$

The above equation is called the secant method. This method now requires two initial guesses, but unlike the bisection method, the two initial guesses do not need to bracket the root of the equation. The secant method is an open method and may or may not converge. However, when secant method converges, it will typically converge faster than the bisection method. However, since the derivative is approximated as given by Equation (2), it typically converges slower than the Newton-Raphson method.

The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, x_{i-1} and x_i, one draws a straight line between $f(x_i)$ and $f(x_{i-1})$ passing through the x-axis at x_{i+1}. ABE and DCE are similar triangles.

Hence

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

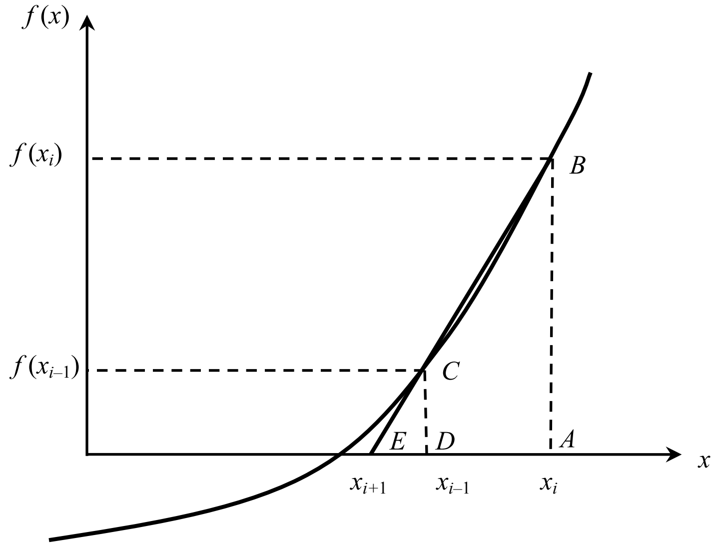

Figure 1 Geometrical representation of the secant method.

Example 1

Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature.

For a 10K3A Betatherm thermistor,

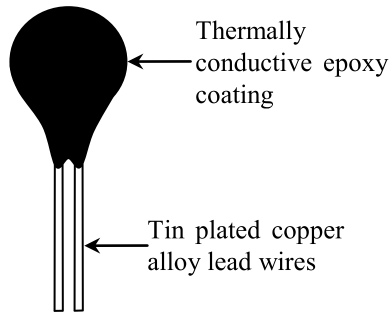

Figure 2 A typical thermistor.

the relationship between the resistance R of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

where T is in Kelvin and R is in ohms.

A thermistor error of no more than $\pm 0.01°C$ is acceptable. To find the range of the resistance that is within this acceptable limit at $19°C$, we need to solve

$$\frac{1}{19.01+273.15} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

and

$$\frac{1}{18.99+273.15} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

Use the secant method of finding roots of equations to find the resistance R at $18.99°C$. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

Solving

$$\frac{1}{18.99+273.15} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

We get

$$f(R) = 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3 - 2.293775\times10^{-3}$$

Let us take the initial guesses of the root of $f(R)=0$ as $R_{-1}=14000$ and $R_0=15000$.

Iteration 1

The estimate of the root is

$$R_1 = R_0 - \frac{f(R_0)(R_0-R_{-1})}{f(R_0)-f(R_{-1})}$$

$$f(R_0) = 2.341077\times10^{-4}\ln(R_0) + 8.775468\times10^{-8}\{\ln(R_0)\}^3 - 2.293775\times10^{-3}$$

$$= 2.341077\times10^{-4}\ln(15000) + 8.775468\times10^{-8}\{\ln(15000)\}^3 - 2.293775\times10^{-3}$$

$$= 3.5383\times 10^{-5}$$

$$f(R_{-1}) = 2.341077\times 10^{-4}\ln(R_{-1}) + 8.775468\times 10^{-8}\{\ln(R_{-1})\}^3 - 2.293775\times 10^{-3}$$

$$= 2.341077\times 10^{-4}\ln(14000) + 8.775468\times 10^{-8}\{\ln(14000)\}^3 - 2.293775\times 10^{-3}$$

$$= 1.7563\times 10^{-5}$$

$$R_1 = 15000 - \frac{(3.5383\times 10^{-5})(15000-14000)}{(3.5383\times 10^{-5}) - (1.7563\times 10^{-5})}$$

$$= 13014$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{R_1 - R_0}{R_1}\right|\times 100$$

$$= \left|\frac{13014-15000}{13014}\right|\times 100$$

$$= 15.257\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of less than 5% for one significant digit to be correct in your result.

<u>Iteration 2</u>

The estimate of the root is

$$R_2 = R_1 - \frac{f(R_1)(R_1 - R_0)}{f(R_1) - f(R_0)}$$

$$f(R_1) = 2.341077\times 10^{-4}\ln(R_1) + 8.775468\times 10^{-8}\{\ln(R_1)\}^3 - 2.293775\times 10^{-3}$$

$$= 2.341077\times 10^{-4}\ln(13014) + 8.775468\times 10^{-8}\{\ln(13014)\}^3 - 2.293775\times 10^{-3}$$

$$= -1.2658\times 10^{-6}$$

$$R_2 = 13014 - \frac{(-1.2658\times 10^{-6})(13014-15000)}{(-1.2658\times 10^{-6}) - (3.5383\times 10^{-5})}$$

$$= 13083$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{R_2 - R_1}{R_2}\right|\times 100$$

$$= \left|\frac{13083-13014}{13083}\right|\times 100$$

$$= 0.52422\%$$

The number of significant digits at least correct is 1, because the absolute relative approximate error is less than 5%.

Iteration 3

The estimate of the root is

$$R_3 = R_2 - \frac{f(R_2)(R_2 - R_1)}{f(R_2) - f(R_1)}$$

$$\begin{aligned} f(R_2) &= 2.341077\times10^{-4}\ln(R_2) + 8.775468\times10^{-8}\{\ln(R_2)\}^3 - 2.293775\times10^{-3} \\ &= 2.341077\times10^{-4}\ln(13083) + 8.775468\times10^{-8}\{\ln(13083)\}^3 - 2.293775\times10^{-3} \\ &= 8.8907\times10^{-8} \end{aligned}$$

$$\begin{aligned} R_3 &= 13083 - \frac{(8.8911\times10^{-8})(13083 - 13014)}{(8.8911\times10^{-8}) - (-1.2658\times10^{-6})} \\ &= 13078 \end{aligned}$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$\begin{aligned} |\in_a| &= \left|\frac{R_3 - R_2}{R_3}\right| \times 100 \\ &= \left|\frac{13078 - 13083}{13078}\right| \times 100 \\ &= 0.034415\% \end{aligned}$$

The number of significant digits at least correct is 3, because the absolute relative approximate error is less than 0.05%.

NONLINEAR EQUATIONS	
Topic	Secant Method-More Examples
Summary	Examples of Secant Method
Major	Electrical Engineering
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