# Newton's Divided Difference Polynomial Method of Interpolation

Electrical Engineering Majors

Authors: Autar Kaw, Jai Paul

http://numericalmethods.eng.usf.edu

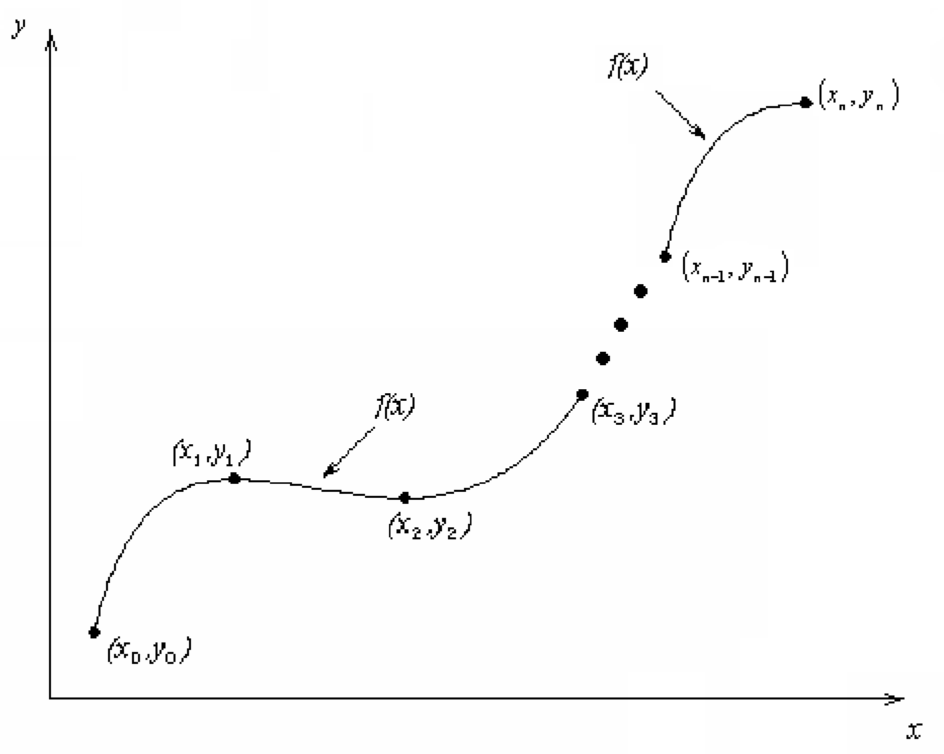
Transforming Numerical Methods Education for STEM Undergraduates

# Newton's Divided Difference Method of Interpolation

http://numericalmethods.eng.usf.edu

# What is Interpolation ?

Given $(x_0,y_0)$, $(x_1,y_1)$, ...... $(x_n,y_n)$, find the value of 'y' at a value of 'x' that is not given.



# Interpolants

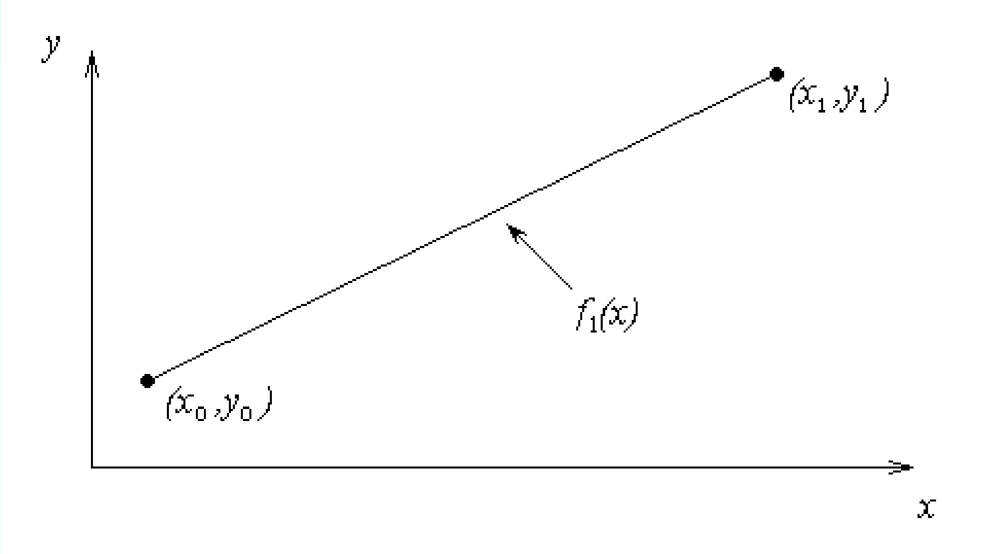
Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Newton's Divided Difference Method

<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$




where

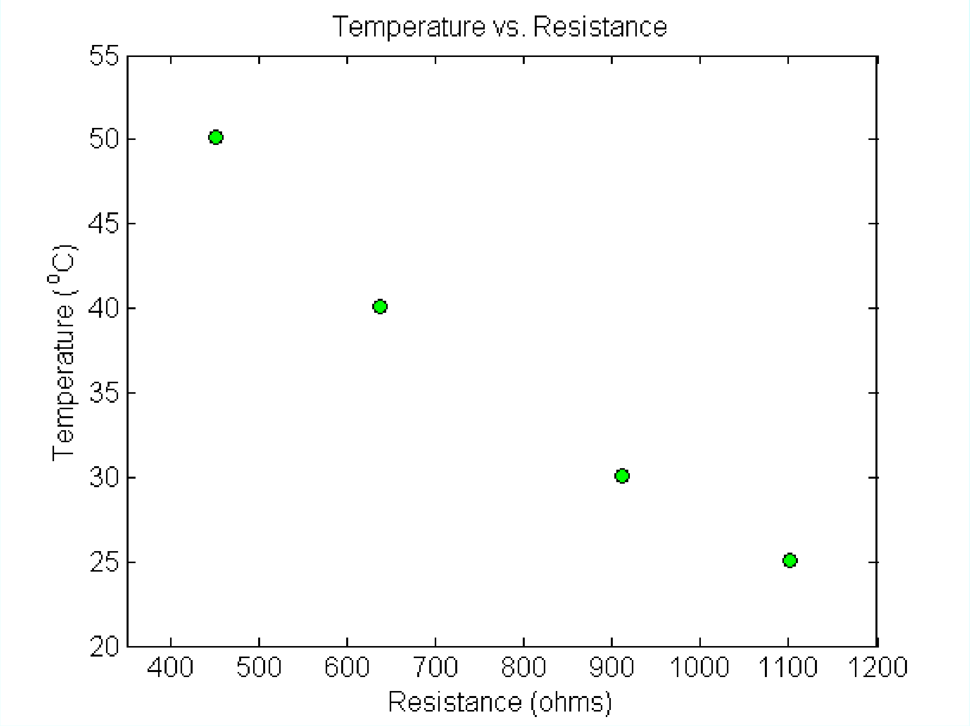
$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

# Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Newton Divided Difference method for linear interpolation.

| R (Ω) | T( C) |
|---|---|
| 1101.0 | 25.113 |
| 911.3 | 30.131 |
| 636.0 | 40.120 |
| 451.1 | 50.128 |



# Linear Interpolation

$$T(R) = b_0 + b_1(R - R_0)$$

$$R_0 = 911.3, T(R_0) = 30.131$$

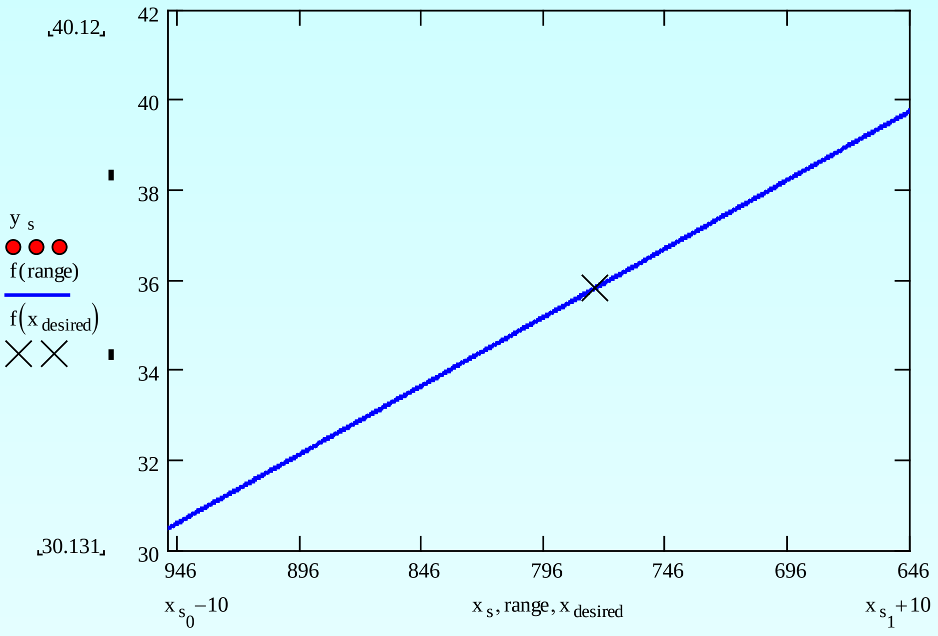
$$R_1 = 636.0, T(R_1) = 40.120$$

$$b_0 = T(R_0)$$

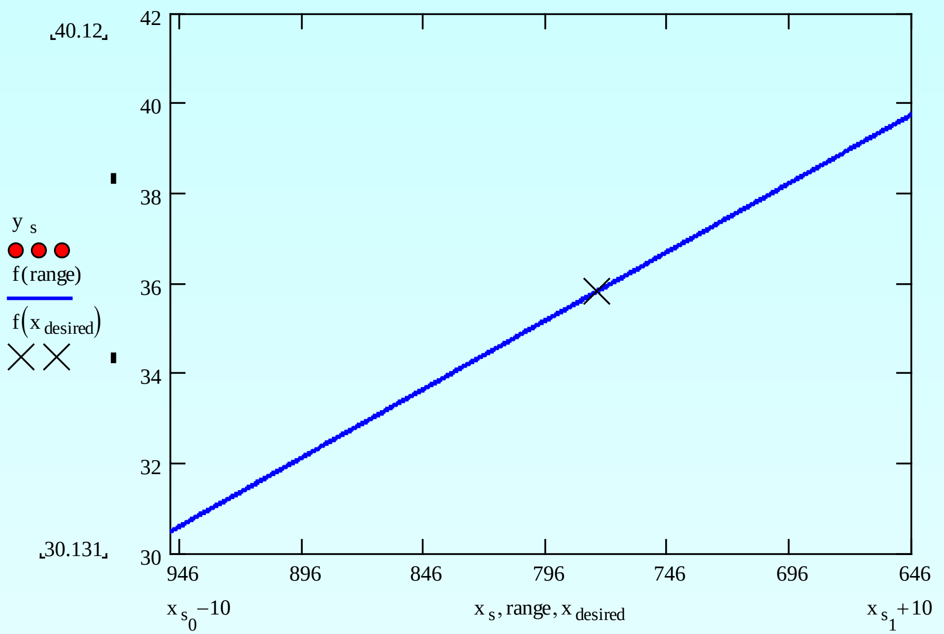
$$= 30.131$$

$$b_1 = \frac{T(R_1) - T(R_0)}{R_1 - R_0} = \frac{40.120 - 30.131}{636.0 - 911.3}$$

$$= -0.036284$$



# Linear Interpolation (contd)




$$T(R) = b_0 + b_1(R - R_0)$$

$$= 30.131 - 0.036284(R - 911.3), \qquad 636.0 \le R \le 911.3$$

At $R = 754.8$

$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$

$$= 35.809°C$$

# Quadratic Interpolation

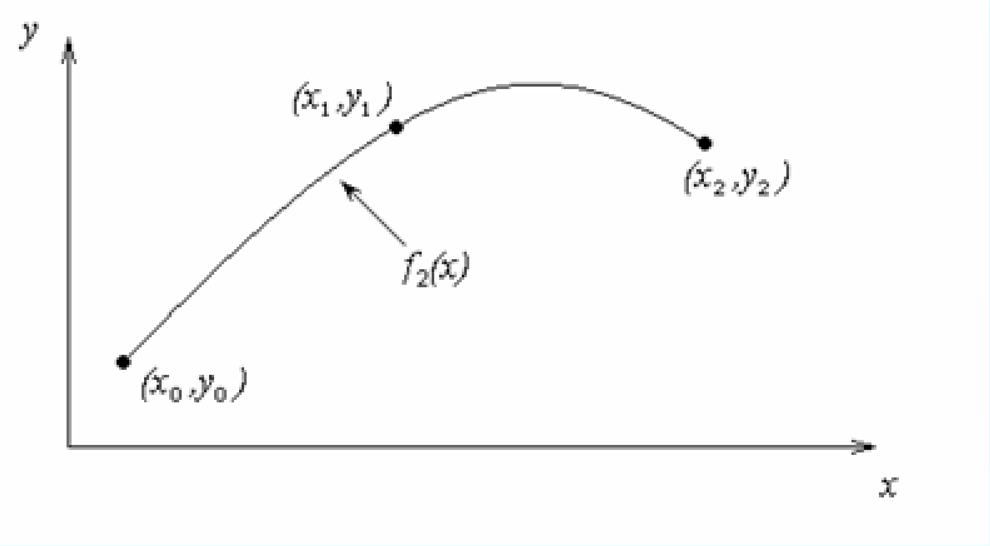
Given $(x_0, y_0)$, $(x_1, y_1)$, and $(x_2, y_2)$, fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

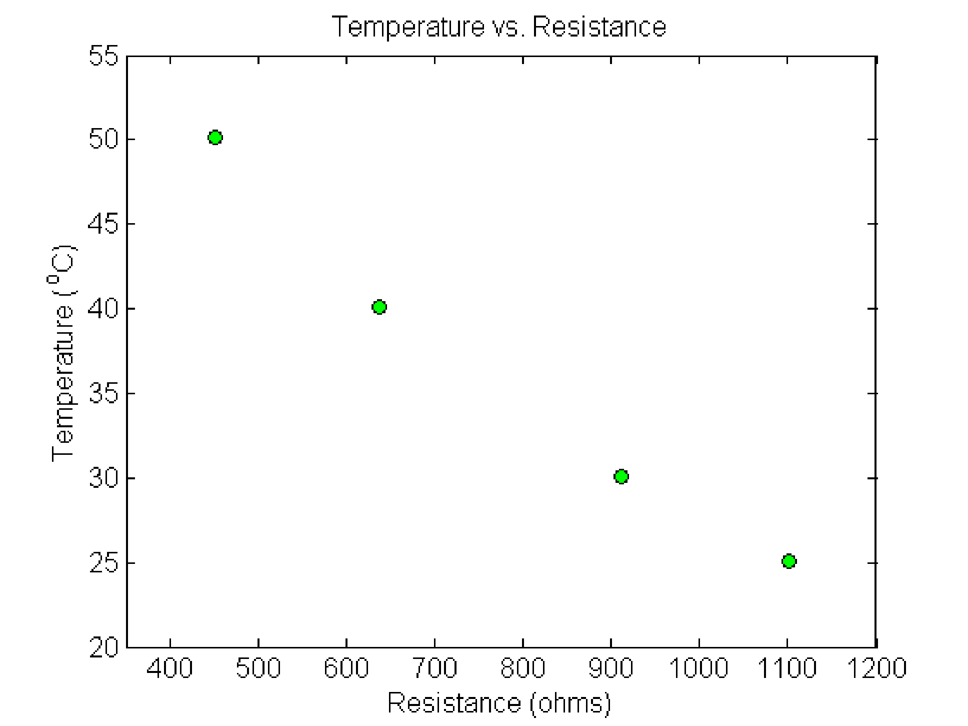
$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



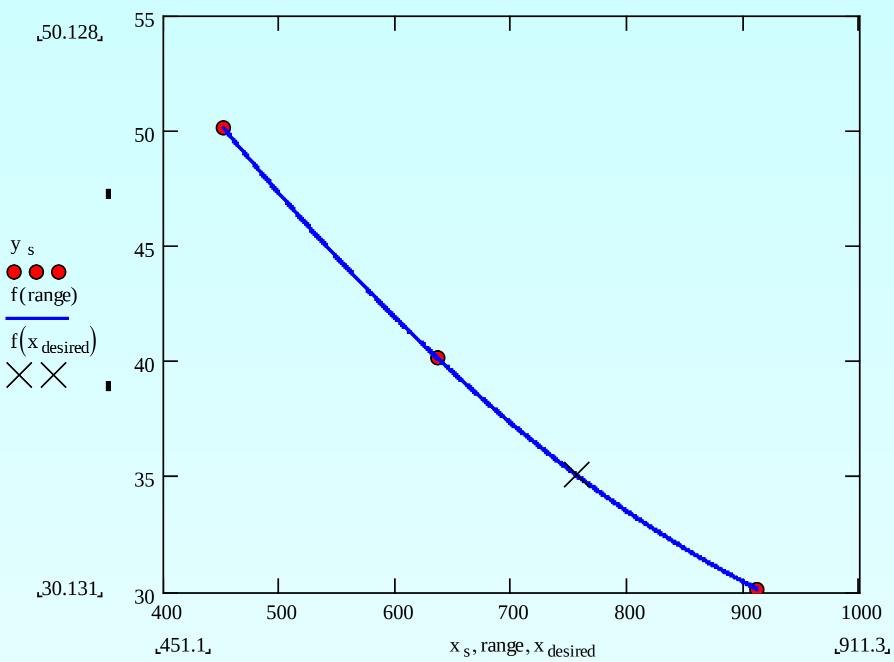
# Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Newton Divided Difference method for quadratic interpolation.

| R (Ω) | T( C) |
|---|---|
| 1101.0 | 25.113 |
| 911.3 | 30.131 |
| 636.0 | 40.120 |
| 451.1 | 50.128 |



# Quadratic Interpolation (contd)




$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1)$$

$$R_0 = 911.3, T(R_0) = 30.131$$

$$R_1 = 636.0, \ T(R_1) = 40.120$$

$$R_2 = 451.1, T(R_2) = 50.128$$

# Quadratic Interpolation (contd)

$$b_0 = T(R_0)$$

$$= 30.131$$

$$b_1 = \frac{T(R_1) - T(R_0)}{R_1 - R_0} = \frac{40.120 - 30.131}{636.0 - 911.3}$$

$$= -0.036284$$

$$b_2 = \frac{\frac{T(R_2) - T(R_1)}{R_2 - R_1} - \frac{T(R_1) - T(R_0)}{R_1 - R_0}}{R_2 - R_0} = \frac{\frac{50.128 - 40.120}{451.1 - 636.0} - \frac{40.120 - 30.131}{636.0 - 911.3}}{451.1 - 911.3}$$

$$= \frac{-0.054127 + 0.036284}{-460.2}$$

$$= 3.8771 \times 10^{-5}$$

# Quadratic Interpolation (contd)

$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1)$$

$$= 30.131 - 0.036284(R - 911.3) + 3.8771 \times 10^{-5}(R - 911.3)(R - 636.0), \; 451.1 \le R \le 911.3$$

At $R = 754.8$,

$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3) + 3.8771 \times 10^{-5}(754.8 - 911.3)(754.8 - 636.0)$$

$$= 35.089°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{35.089 - 35.809}{35.089}\right| \times 100$$

$$= 2.0543\%$$

# General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

# General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

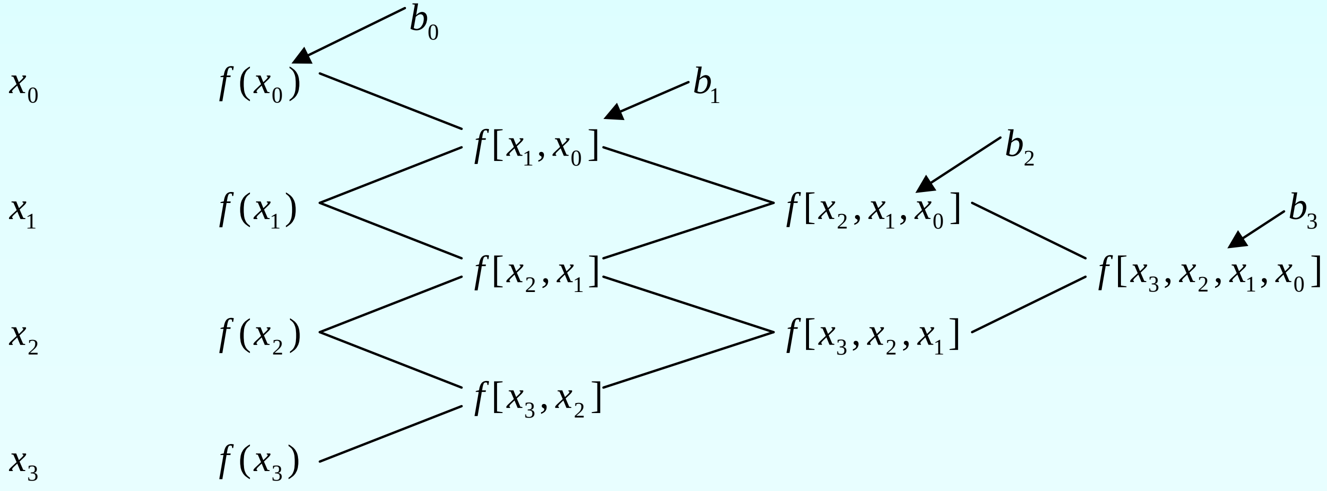
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

# General form

The third order polynomial, given $(x_0, y_0)$, $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$, is

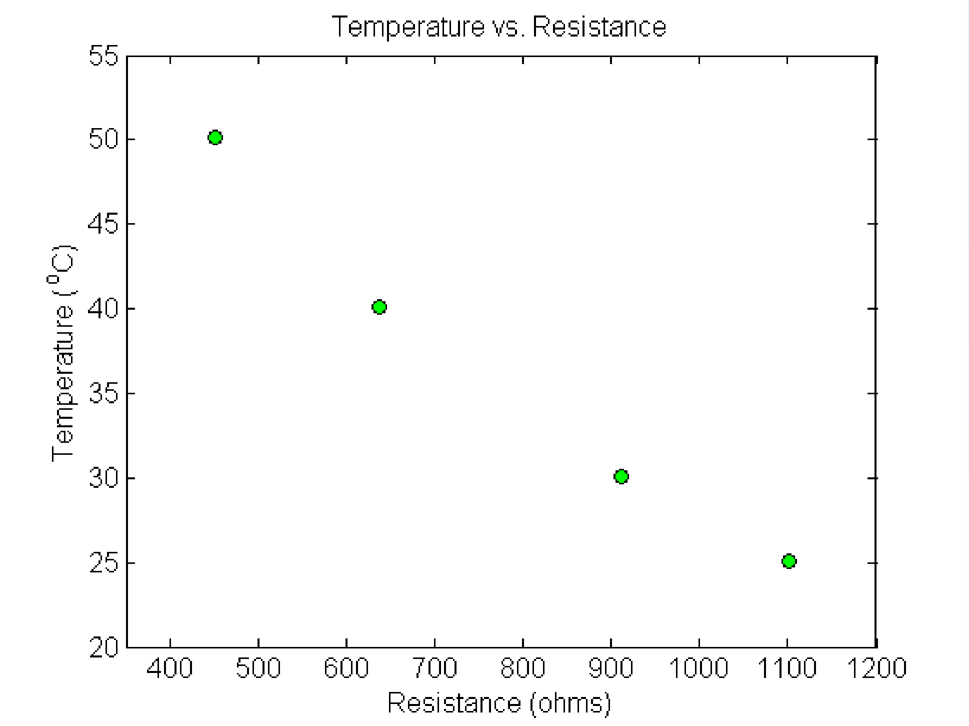
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



# Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Newton Divided Difference method for cubic interpolation.

| R (Ω) | T( C) |
|---|---|
| 1101.0 | 25.113 |
| 911.3 | 30.131 |
| 636.0 | 40.120 |
| 451.1 | 50.128 |



# Example

For the third order polynomial, we choose the temperature given by

$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2)$$

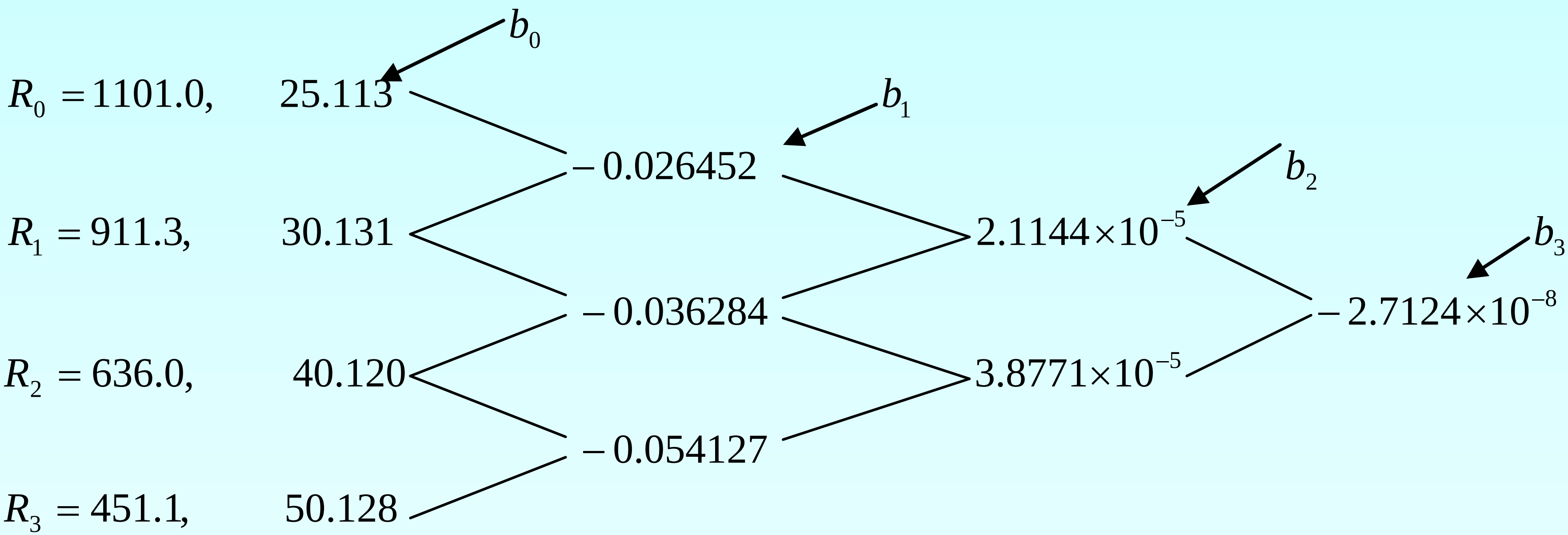
$$R_0 = 1101.0, \quad T(R_0) = 25.113$$

$$R_1 = 911.3, \quad T(R_1) = 30.131$$

$$R_2 = 636.0, \quad T(R_2) = 40.120$$

$$R_3 = 451.1, \quad T(R_3) = 50.128$$

# Example




The values of the constants are found as:

$$b_0 = 25.113 \qquad b_1 = -0.026452 \qquad b_2 = 2.1144\times10^{-5} \qquad b_3 = -2.7124\times10^{-8}$$

# Example

$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2)$$

$$= 25.113 - 0.026452(R - 1101.0) + 2.1144\times10^{-5}(R - 1101.0)(R - 911.3)$$

$$- 2.7124\times10^{-8}(R - 1101.0)(R - 911.3)(R - 636.0), \quad 451.1 \le R \le 1101.0$$

At $R = 754.8,$

$$T(754.8) = 25.113 - 0.026452(754.8 - 1101.0) + 2.1144\times10^{-5}(754.8 - 1101.0)(754.8 - 911.3)$$

$$- 2.7124\times10^{-8}(754.8 - 1101.0)(754.8 - 911.3)(754.8 - 626.0)$$

$$= 35.242^\circ\text{C}$$

The absolute percentage relative approximate error, $\left|\in_a\right|$ for the value obtained for T(754.8)

between second and third order polynomial is

$$\left|\in_a\right| = \left|\frac{35.242 - 35.089}{35.242}\right| \times 100$$

$$= 0.43458\%$$

# Comparison Table

| Order of Polynomial | 1 | 2 | 3 |
|---|---|---|---|
| Temperature $°C$ | 35.809 | 35.089 | 35.242 |
| Absolute Relative Approximate Error | ---------- | 2.0543% | 0.43458% |

# Actual Calibration

The actual calibration curve used by industry is given by

$$\frac{1}{T} = b_0 + b_1(\ln R - \ln R_0) + b_2(\ln R - \ln R_0)(\ln R - \ln R_1) + b_3(\ln R - \ln R_0)(\ln R - \ln R_1)(\ln R - \ln R_2)$$

substituting $y = \frac{1}{T}$, and $x = \ln R$, the calibration curve is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

| $R$ (Ω) | $T$ ( C) | $x(\ln R)$ | $y(1/T)$ |
|---|---|---|---|
| 1101.0 | 25.113 | 7.0040 | 0.039820 |
| 911.3 | 30.131 | 6.8149 | 0.033188 |
| 636.0 | 40.120 | 6.4552 | 0.024925 |
| 451.1 | 50.128 | 6.1117 | 0.019949 |

Find the calibration curve and find the temperature corresponding to 754.8 ohms. What is the difference between the results from cubic interpolation? In which method is the difference larger, if the actual measured value at 754.8 ohms is 35.285 $^\circ C$ ?

# Actual Calibration

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

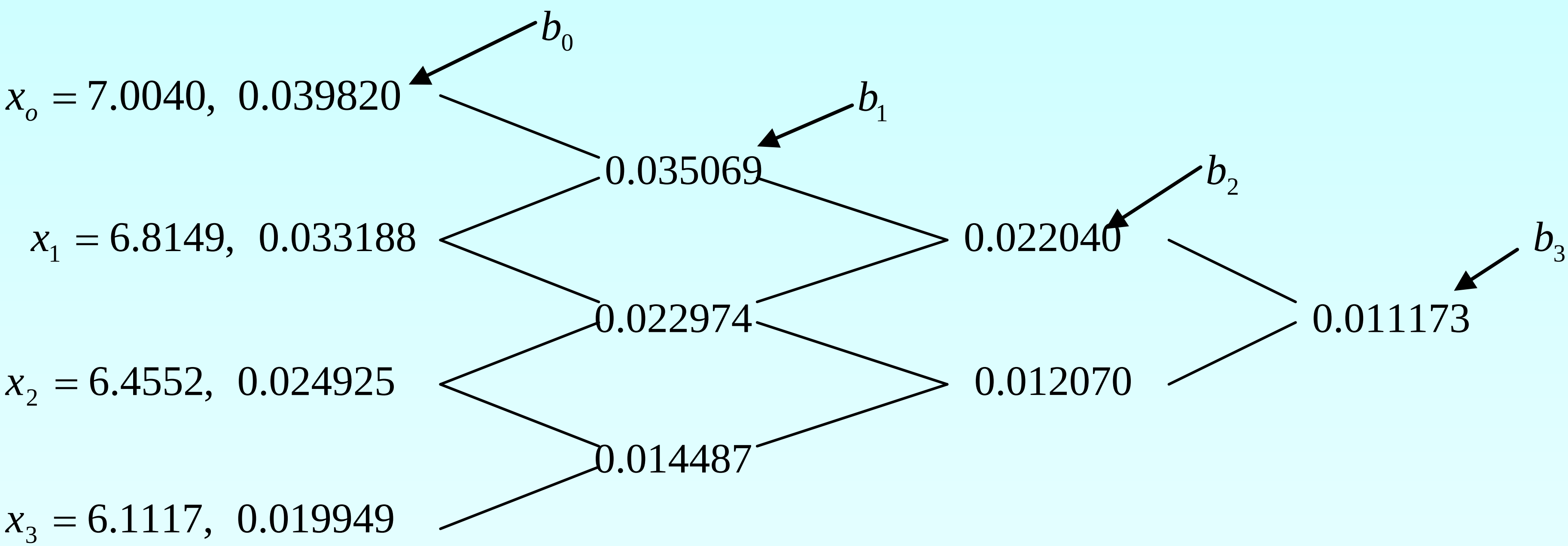
$$x_o = 7.0040, \quad y(x_o) = 0.039820$$

$$x_1 = 6.8149, \quad y(x_1) = 0.033188$$

$$x_2 = 6.4552, \quad y(x_2) = 0.024925$$

$$x_3 = 6.1117, \quad y(x_3) = 0.019949$$

# Actual Calibration




The values of the constants are found as:

$$b_0 = 0.039820 \qquad b_1 = 0.035069$$

$$b_2 = 0.022040 \qquad b_3 = 0.011173$$

# Actual Calibration

$$y(x)=b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)+b_3(x-x_0)(x-x_1)(x-x_2)$$
$$=0.039820+0.035069(x-7.0040)+0.022974(x-7.0040)(x-6.8149)$$
$$+0.011173(x-7.0040)(x-6.8149)(x-6.4552),\ \ 6.1117\le x\le 7.0040$$

Since $x=\ln 754.8=6.6265$

At $x=6.6265$,

$$y(6.6265)=0.039820+0.035071(6.6265-7.0040)+0.022972(6.6265-7.0040)(6.6265-6.8149)$$
$$+0.011182(6.6265-7.0040)(6.6265-6.8149)(6.6265-6.4552)$$

$$=0.028285$$

$$T=\frac{1}{y}\qquad=\frac{1}{0.028285}$$

$$=35.355°C$$

# Actual Calibration

Since the actual measured value at 754.8 ohms is $35.285\,°C$, the absolute relative true error between the value used for Cubic Interpolation is

$$|\in_t| = \left|\frac{35.285 - 35.242}{35.285}\right| \times 100$$

$$= 0.12253\%$$

and for actual calibration is

$$|\in_t| = \left|\frac{35.285 - 35.355}{35.285}\right| \times 100$$

$$= 0.19825\%$$

Therefore, the calibration curve obtained more accurate results than a cubic polynomial interpolant given by Newton's Divided Difference method

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

# THE END

http://numericalmethods.eng.usf.edu