# Spline Interpolation Method

Electrical Engineering Majors

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# Spline Method of Interpolation

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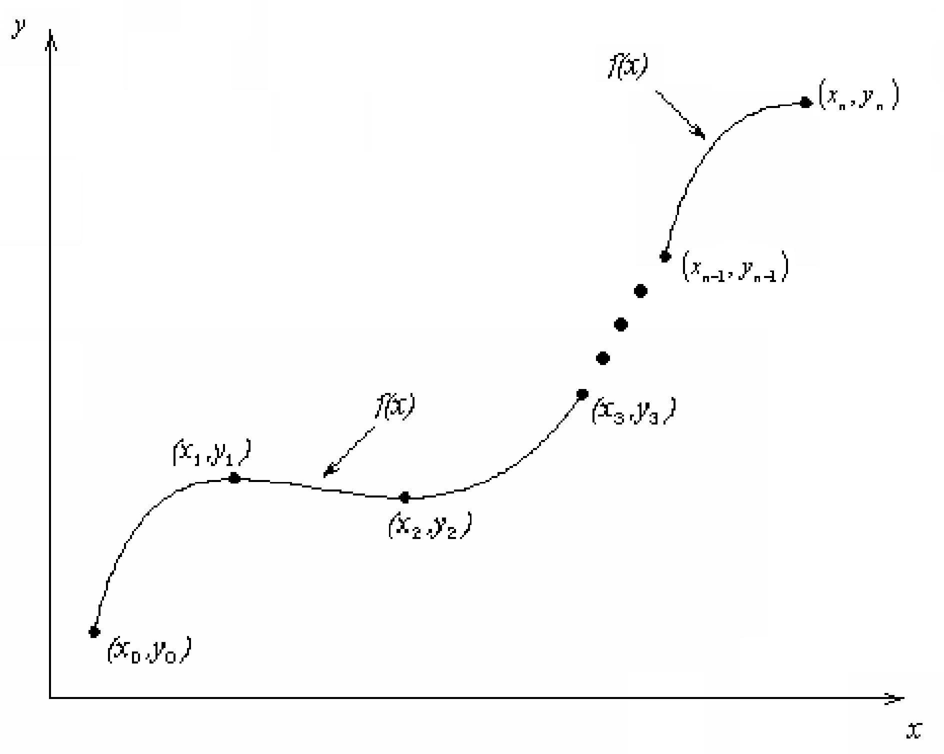
# What is Interpolation ?

Given $(x_0,y_0)$, $(x_1,y_1)$, ...... $(x_n,y_n)$, find the value of 'y' at a value of 'x' that is not given.



# Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

**Table : Six equidistantly spaced points in [-1, 1]**

| $x$ | $y = \frac{1}{1+25x^2}$ |
|---|---|
| -1.0 | 0.038461 |
| -0.6 | 0.1 |
| -0.2 | 0.5 |
| 0.2 | 0.5 |
| 0.6 | 0.1 |
| 1.0 | 0.038461 |

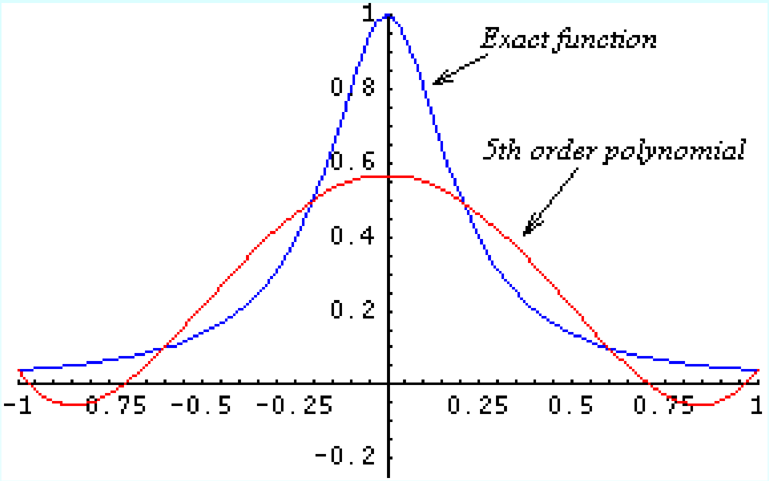



**Figure : 5$^{th}$ order polynomial vs. exact function**

# Why Splines ?

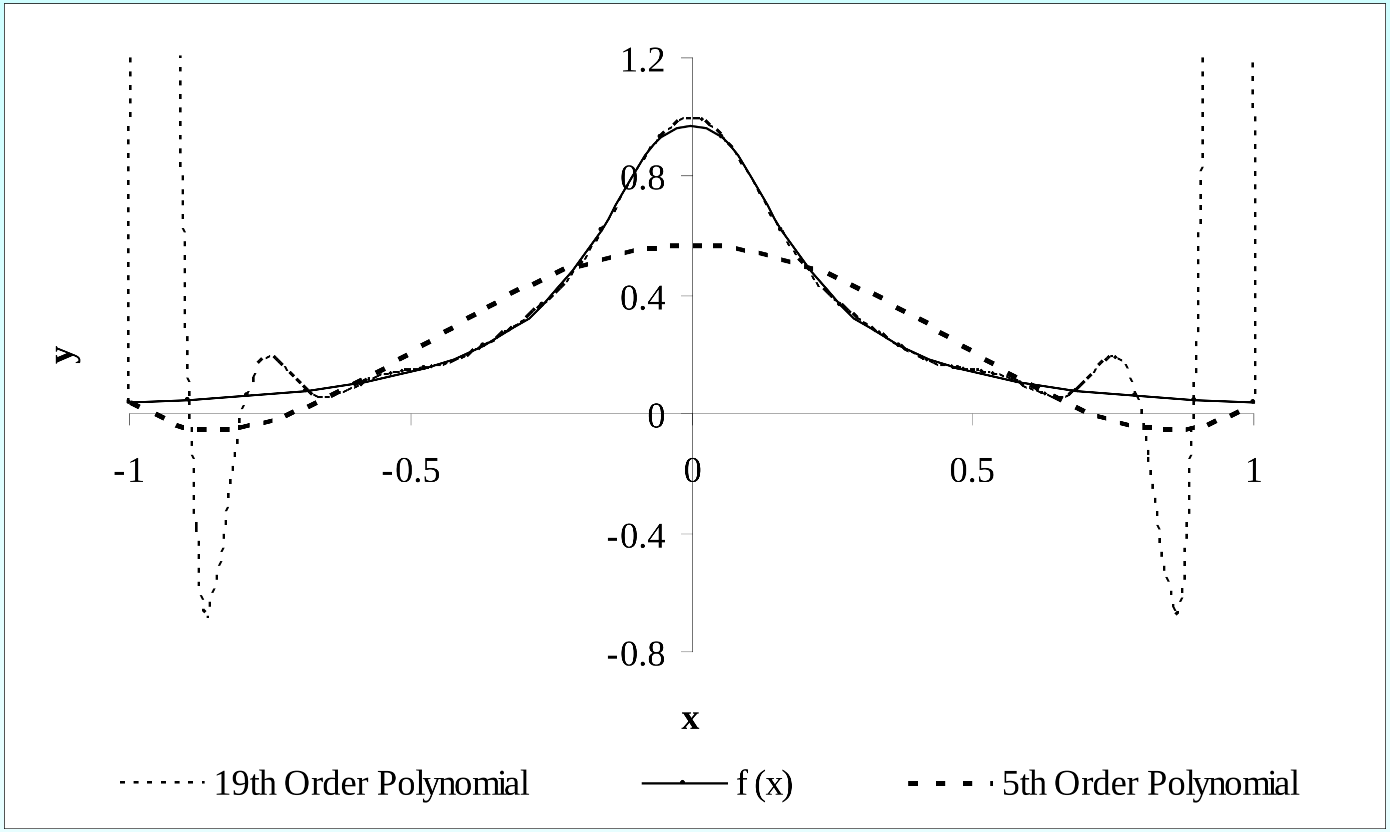
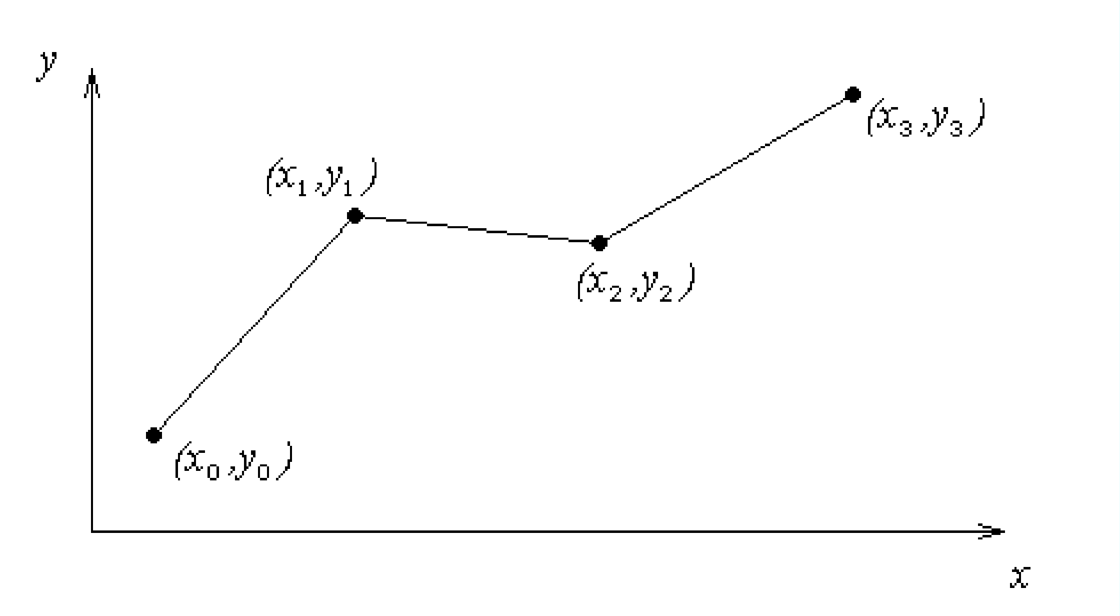



**Figure : Higher order polynomial interpolation is a bad idea**

# Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), ......, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

**Figure : Linear splines**



# Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

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$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

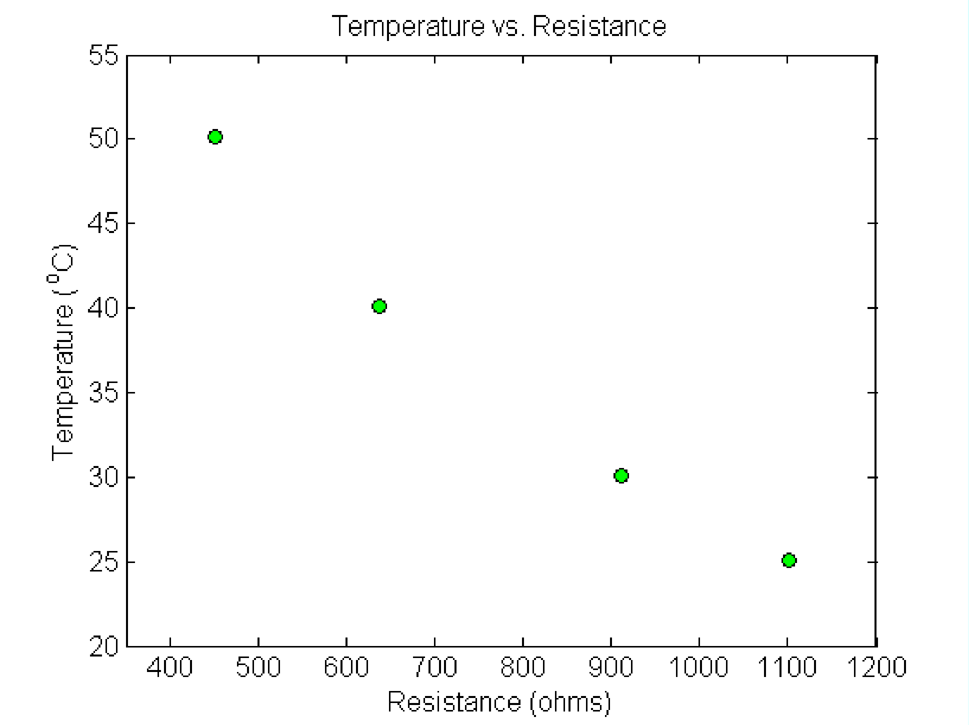
$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between $x_{i-1}$ and $x_i$.

# Example

Thermistors are based on materials' change in resistance with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using linear spline interpolation.

| R (Ω) | T( C) |
|---|---|
| 1101.0 | 25.113 |
| 911.3 | 30.131 |
| 636.0 | 40.120 |
| 451.1 | 50.128 |



# Linear Interpolation

$$R_0 = 911.3, \qquad T(R_0) = 30.131$$

$$R_1 = 636.0, \qquad T(R_1) = 40.120$$

$$T(R) = T(R_0) + \frac{T(R_1) - T(R_0)}{R_1 - R_0}(R - R_0)$$

$$= 30.131 + \frac{40.120 - 30.131}{636.0 - 911.3}(R - 911.3)$$

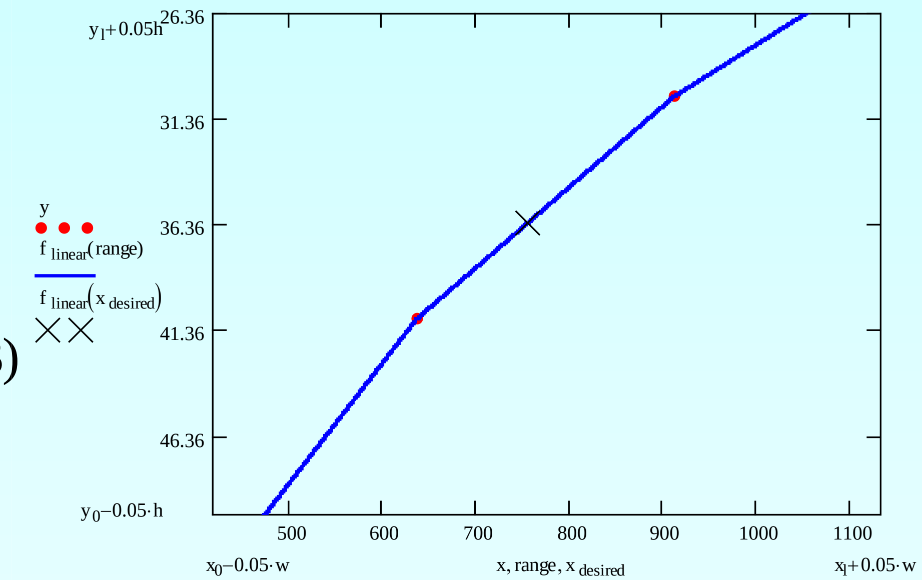
$$636.0 \le R \le 911.3$$

$$T(R) = 30.131 - 0.036284(R - 911.3)$$

At $R = 754.8$,

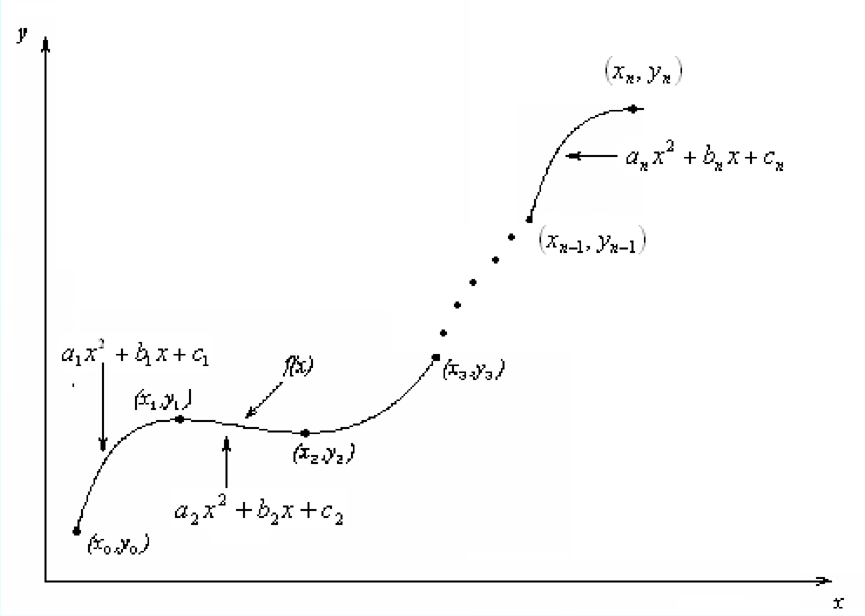
$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$

$$= 35.809°C$$



# Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by




$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&\;. \\
&\;. \\
&\;. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$

Find $a_i, b_i, c_i, i = 1, 2, \ldots, \mathrm{n}$

# Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

$$\vdots$$

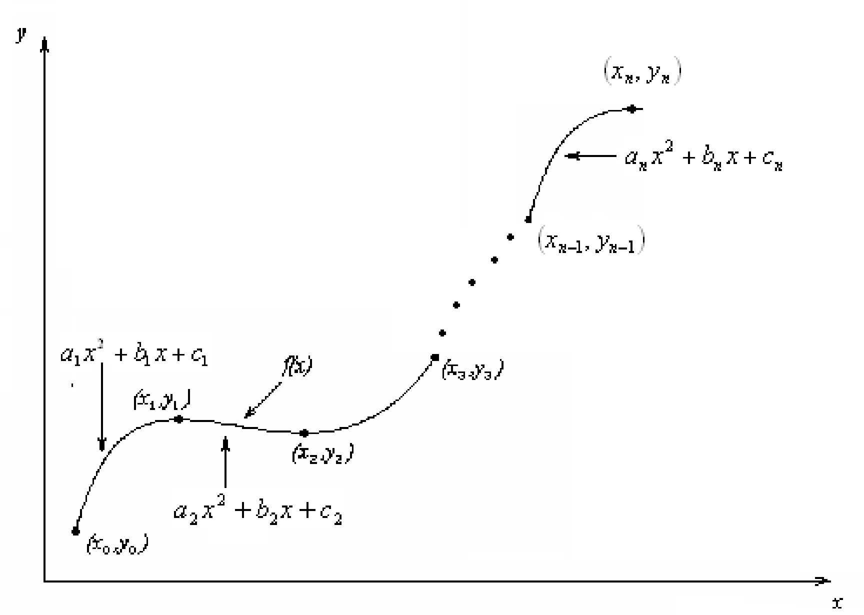
$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

$$\vdots$$

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$




This condition gives 2n equations

# Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

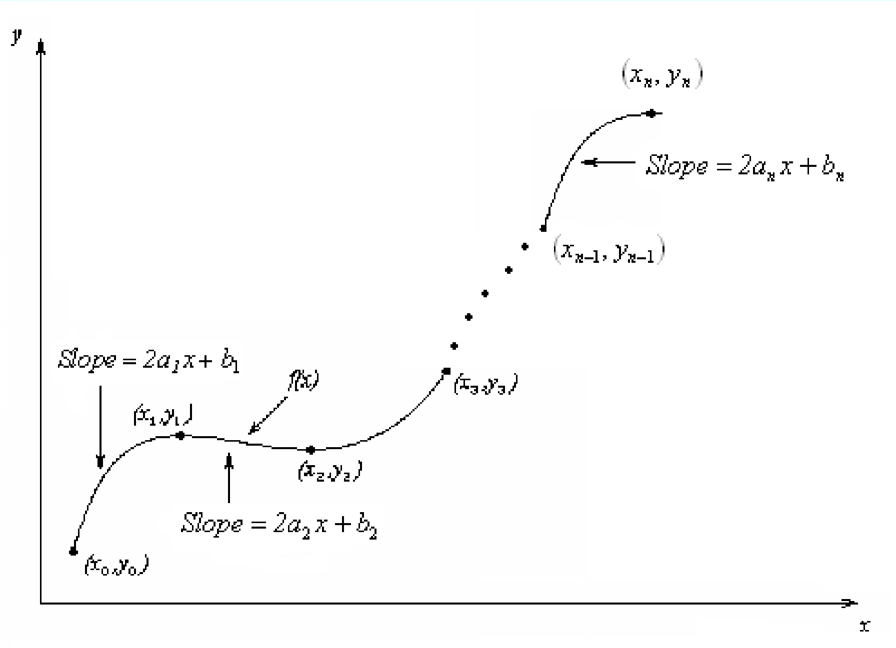
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



# Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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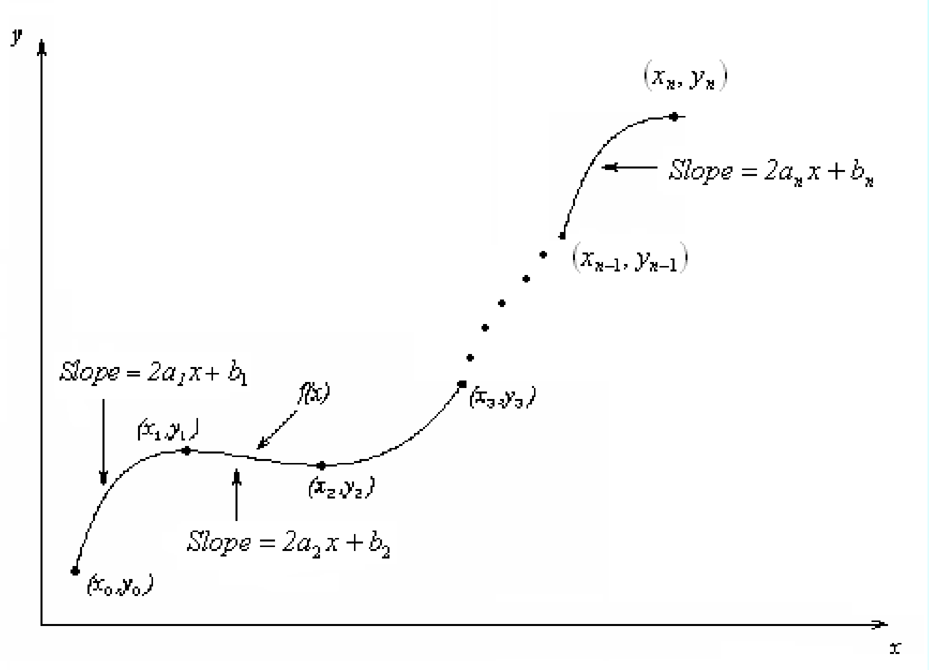
$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$




We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

# Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$f(x) = a_1 x^2 + b_1 x + c_1, \qquad x_0 \le x \le x_1$$

$$= a_2 x^2 + b_2 x + c_2, \qquad x_1 \le x \le x_2$$

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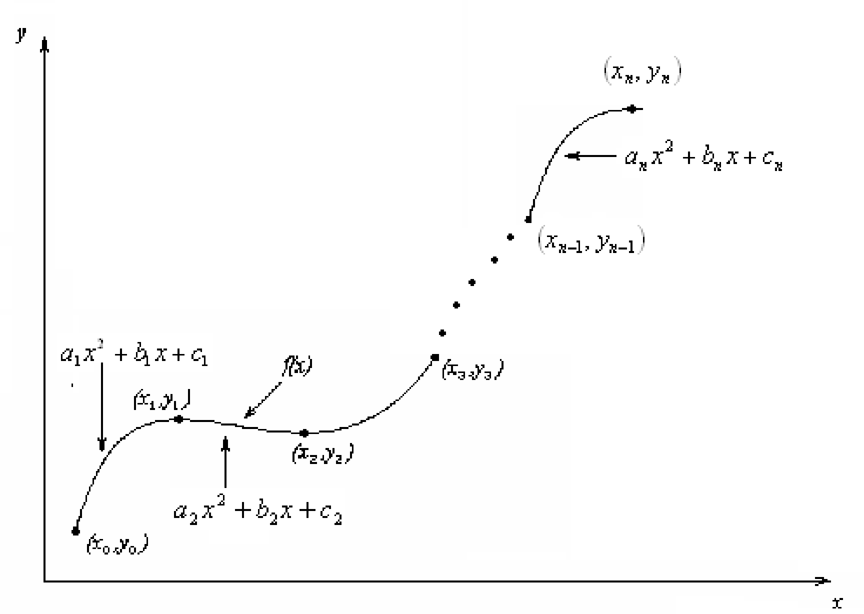
$$= a_n x^2 + b_n x + c_n, \qquad x_{n-1} \le x \le x_n$$



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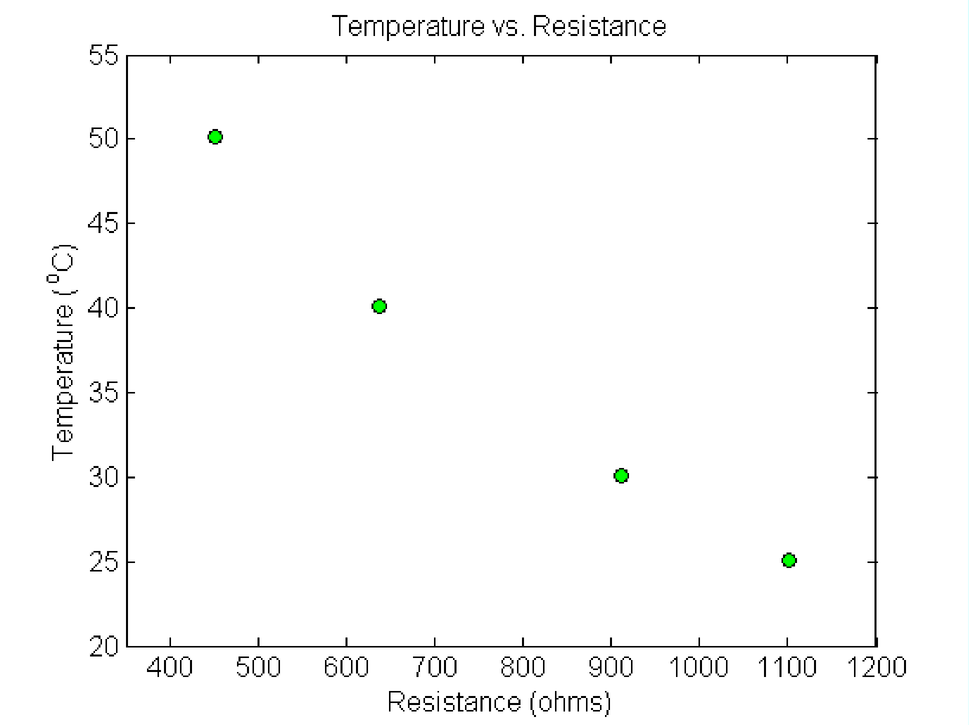


# Solution

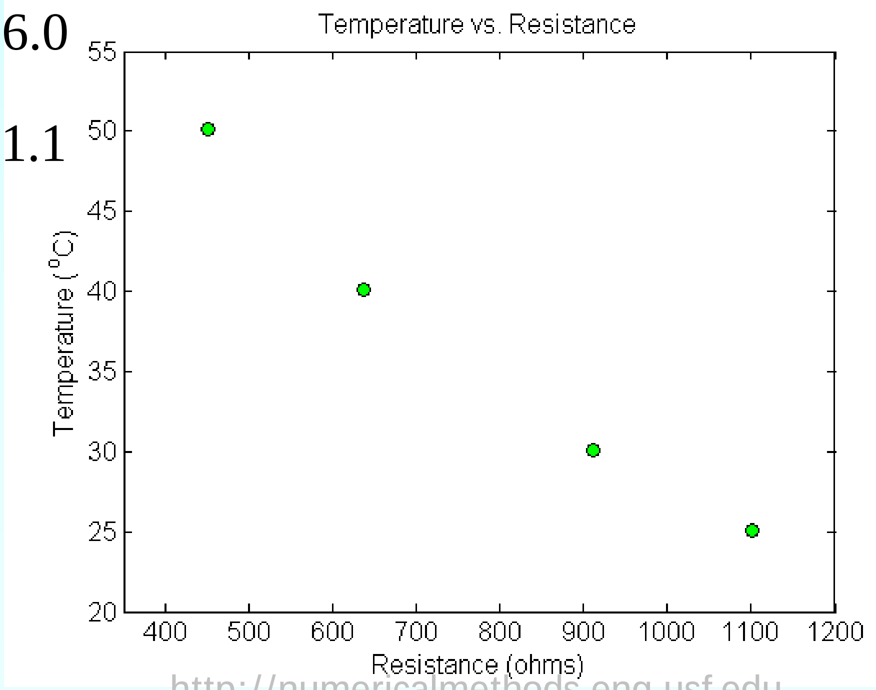
Since there are four data points,

three quadratic splines pass through them.

$$T(R) = a_1R^2 + b_1R + c_1, \qquad 1101.0 \le R \le 911.3$$

$$= a_2R^2 + b_2R + c_2, \qquad 911.3 \le R \le 636.0$$

$$= a_3R^2 + b_3R + c_3, \qquad 636.0 \le R \le 451.1$$



# Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1R^2 + b_1R + c_1$ passes through R = 1101.0 and R = 911.3,

$$a_1(1101.0)^2 + b_1(1101.0) + c_1 = 25.113 \qquad (1)$$

$$a_1(911.3)^2 + b_1(911.3) + c_1 = 30.131 \qquad (2)$$

Similarly,

$$a_2(911.3)^2 + b_2(911.3) + c_2 = 30.131 \qquad (3)$$

$$a_2(636.0)^2 + b_2(636.0) + c_2 = 40.120 \qquad (4)$$

$$a_3(636.0)^2 + b_3(636.0) + c_3 = 40.120 \qquad (5)$$

$$a_3(451.1)^2 + b_3(451.1) + c_3 = 50.128 \qquad (6)$$

# Solution (contd)

Quadratic splines have continuous

derivatives at the interior data points

At R = 911.3

$$2a_1(911.3) + b_1 - 2a_2(911.3) - b_2 = 0 \qquad (7)$$

At R = 636.0

$$2a_2(636.0) + b_2 - 2a_3(636.0) - b_3 = 0 \qquad (8)$$

Assuming the first spline $a_1R^2 + b_1R + c_1$ is linear,

$$a_1 = 0 \qquad (9)$$

# Solution (contd)

$$\begin{bmatrix} 1.2122\times10^{6} & 1101.0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8.3047\times10^{5} & 911.3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.3047\times10^{5} & 911.3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.0450\times10^{5} & 636.0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.0450\times10^{5} & 636.0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0349\times10^{5} & 451.1 & 1 \\ 1822.6 & 1 & 0 & -1822.6 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1272 & 1 & 0 & -1272 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25.113 \\ 30.131 \\ 30.131 \\ 40.120 \\ 40.120 \\ 50.128 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Solution (contd)

Solving the above 9 equations gives the 9 unknowns as

| $i$ | $a_i$ | $b_i$ | $c_i$ |
|---|---|---|---|
| 1 | 0 | $-0.026452$ | 54.237 |
| 2 | $3.5713\times10^{-5}$ | $-0.091543$ | 83.895 |
| 3 | $4.3325\times10^{-5}$ | $-0.10122$ | 86.974 |

# Solution (contd)

Therefore, the splines are given by

$$T(R) = -0.026452R + 54.237, \qquad 911.3 \le R \le 1101.0$$

$$= 3.5713\times10^{-5}R^2 - 0.091543R + 83.895, \qquad 636.0 \le R \le 911.3$$

$$= 4.3325\times10^{-5}R^2 - 0.10122R + 86.974, \qquad 451.1 \le R \le 636.0$$

At R = 754.8

$$T(754.8) = 3.5713\times10^{-5}(754.8)^2 - 0.091543(754.8) + 83.895$$

$$= 35.145°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic splines is

$$|\in_a| = \left|\frac{35.145 - 35.809}{35.145}\right| \times 100$$

$$= 1.8892\%$$

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/spline_method.html

# THE END

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