Chapter 05.05
Spline Method of Interpolation

After reading this chapter, you should be able to:
1. *interpolate data using spline interpolation, and*
2. *understand why spline interpolation is important.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to
   (A) evaluate,
   (B) differentiate, and
   (C) integrate
relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. Several methods to obtain such a polynomial include the direct method, Newton's divided difference polynomial method and the Lagrangian interpolation method.

So is the spline method yet another method of obtaining this n^{th} order polynomial. ...... NO! Actually, when n becomes large, in many cases, one may get oscillatory behavior in the resulting polynomial. This was shown by Runge when he interpolated data based on a simple function of

$$y = \frac{1}{1 + 25x^2}$$

on an interval of $[-1, 1]$. For example, take six equidistantly spaced points in $[-1, 1]$ and find y at these points as given in Table 1.
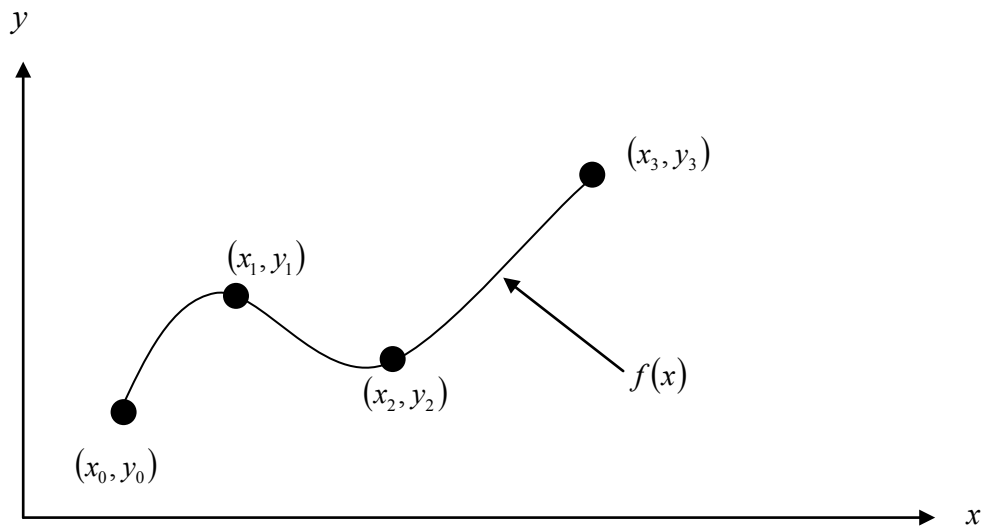
Figure 1   Interpolation of discrete data.

Table 1 Six equidistantly spaced points in [−1, 1].

x	$y = \dfrac{1}{1+25x^2}$
−1.0	0.038461
−0.6	0.1
−0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

Now through these six points, one can pass a fifth order polynomial

$$f_5(x) = 3.1378\times10^{-11}x^5 + 1.2019x^4 - 3.3651\times10^{-11}x^3 - 1.7308x^2 + 1.0004\times10^{-11}x + 5.6731\times10^{-1},$$
$$-1 \leq x \leq 1$$

through the six data points. On plotting the fifth order polynomial (Figure 2) and the original function, one can see that the two do not match well. One may consider choosing more points in the interval [−1, 1] to get a better match, but it diverges even more (see Figure 3), where 20 equidistant points were chosen in the interval [−1, 1] to draw a 19th order polynomial. In fact, Runge found that as the order of the polynomial becomes infinite, the polynomial diverges in the interval of $-1 < x < -0.726$ and $0.726 < x < 1$.

So what is the answer to using information from more data points, but at the same time keeping the function true to the data behavior? The answer is in spline interpolation. The most common spline interpolations used are linear, quadratic, and cubic splines.
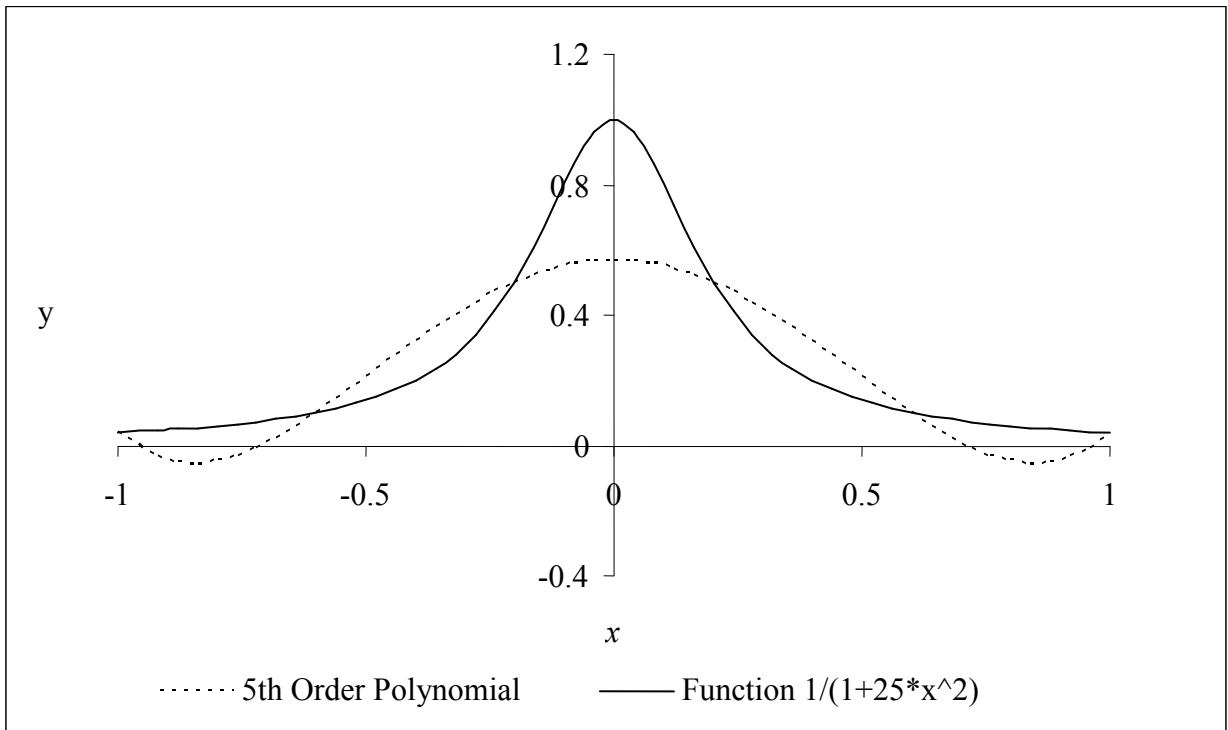
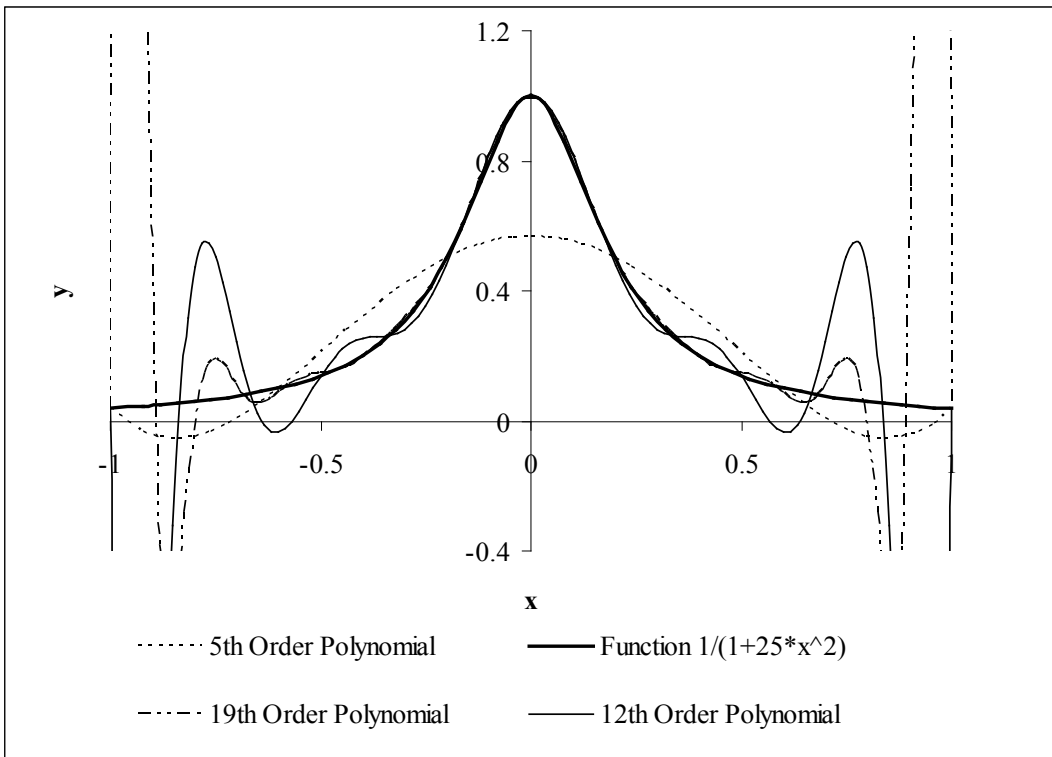**Figure 2** 5th order polynomial interpolation with six equidistant points.

Figure 3 Higher order polynomial interpolation is a bad idea.

Linear Spline Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines (Figure 4) to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $y_i = f(x_i)$.
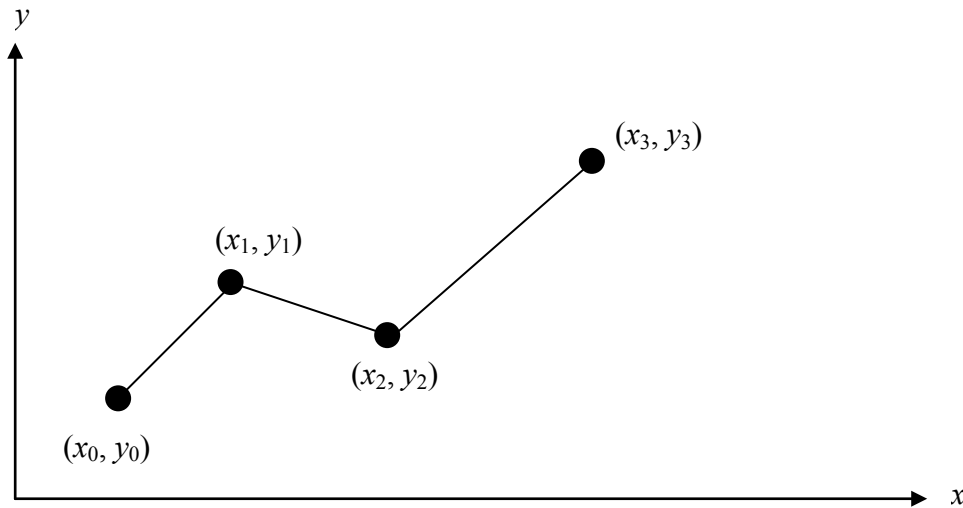

Figure 4   Linear splines.

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

$$\vdots$$

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example 1

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 2.

Table 2 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128
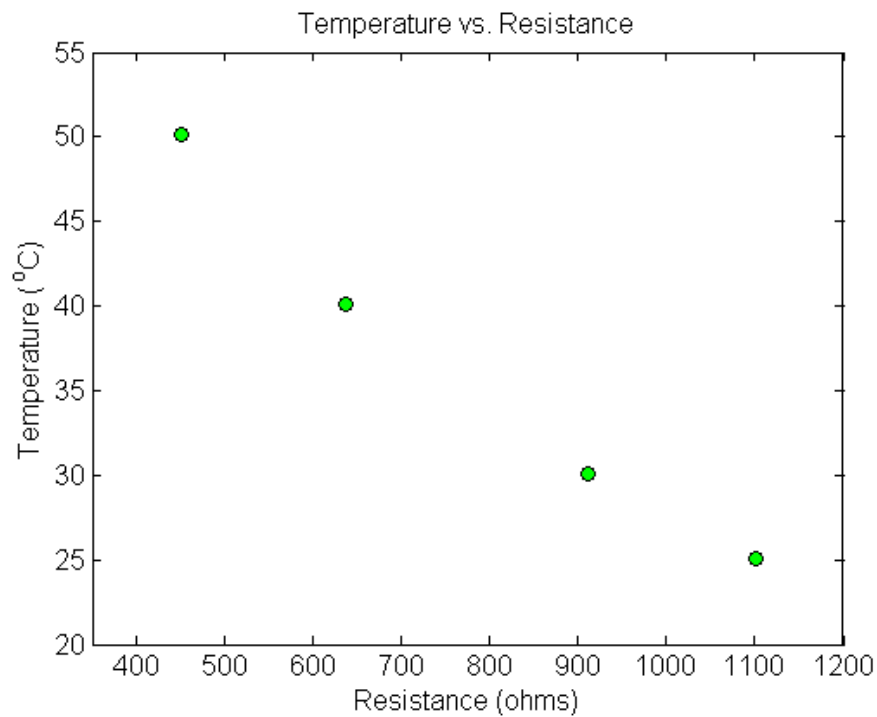
Figure 5 Resistance vs. temperature.

Determine the temperature corresponding to 754.8 ohms using linear splines.

Solution

Since we want to find the temperature at $R = 754.8$ and we are using linear splines, we need to choose the two data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The two points are $R_0 = 911.3$ and $R_1 = 636.0$.

Then

$$R_0 = 911.3, \quad T(R_0) = 30.131$$
$$R_1 = 636.0, \quad T(R_1) = 40.120$$

given

$$T(R) = T(R_0) + \frac{T(R_1) - T(R_0)}{R_1 - R_0}(R - R_0)$$

$$= 30.131 + \frac{40.120 - 30.131}{636.0 - 911.3}(R - 911.3), \quad 636.0 \le R \le 911.3$$

Hence

$$T(R) = 30.131 - 0.036284(R - 911.3)$$

At $R = 754.8$,

$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$
$$= 35.809 \ °C$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Quadratic Splines

In these splines, a quadratic polynomial approximates the data between two consecutive data points. Given $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$f(x) = a_1 x^2 + b_1 x + c_1, \qquad x_0 \le x \le x_1$$
$$= a_2 x^2 + b_2 x + c_2, \qquad x_1 \le x \le x_2$$
$$.$$
$$.$$
$$.$$
$$= a_n x^2 + b_n x + c_n, \qquad x_{n-1} \le x \le x_n$$

So how does one find the coefficients of these quadratic splines? There are $3n$ such coefficients

$$a_i, \ i = 1, 2, \ldots, n$$
$$b_i, \ i = 1, 2, \ldots, n$$
$$c_i, \ i = 1, 2, \ldots, n$$

To find $3n$ unknowns, one needs to set up $3n$ equations and then simultaneously solve them. These $3n$ equations are found as follows.
1. Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$
$$.$$
$$.$$
$$.$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$
$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$
$$.$$
$$.$$
$$.$$
$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$
$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$

This condition gives $2n$ equations as there are n quadratic splines going through two consecutive data points.

2. The first derivatives of two quadratic splines are continuous at the interior points. For example, the derivative of the first spline

$$a_1 x^2 + b_1 x + c_1$$

is

$$2a_1 x + b_1$$

The derivative of the second spline

$$a_2 x^2 + b_2 x + c_2$$

is

$$2a_2 x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2$$
$$2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

Similarly at the other interior points,

$$2a_2 x_2 + b_2 - 2a_3 x_2 - b_3 = 0$$
$$.$$
$$.$$
$$.$$
$$2a_i x_i + b_i - 2a_{i+1} x_i - b_{i+1} = 0$$
$$.$$
$$.$$
$$.$$
$$2a_{n-1} x_{n-1} + b_{n-1} - 2a_n x_{n-1} - b_n = 0$$

Since there are $(n-1)$ interior points, we have $(n-1)$ such equations. So far, the total number of equations is $(2n) + (n-1) = (3n-1)$ equations. We still then need one more equation.

We can assume that the first spline is linear, that is

$$a_1 = 0$$

This gives us $3n$ equations and $3n$ unknowns. These can be solved by a number of techniques used to solve simultaneous linear equations.

Example 2

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 3.

Table 3 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

a) Determine the temperature corresponding to 754.8 ohms using quadratic splines. Find the absolute relative approximate error for the quadratic approximation.

b) The actual calibration curve used by industry is given by

$$\frac{1}{T} = a_0 + a_1[\ln R] + a_2[\ln R]^2$$

substituting $y = \frac{1}{T}$, and $x = \ln R$,

the calibration curve is given by

$$y = a_0 + a_1 x + a_2 x^2$$

Table 3 Manipulation for the given data.

R (ohm)	T (°C)	x $(\ln R)$	$y\left(\dfrac{1}{T}\right)$
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949

Find the calibration curve using quadratic splines, and find the temperature corresponding to 754.8 ohms. What is the difference between the results from part (a)? Is the difference larger using the results from part (a) or part (b), if the actual measured value at 754.8 ohms is 35.285 °C ?

Solution

a) Since there are four data points, three quadratic splines pass through them.

$$T(R) = a_1 R^2 + b_1 R + c_1, \qquad 1101.0 \le R \le 911.3$$
$$= a_2 R^2 + b_2 R + c_2, \qquad 911.3 \le R \le 636.0$$
$$= a_3 R^2 + b_3 R + c_3, \qquad 636.0 \le R \le 451.1$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1 R^2 + b_1 R + c_1$ passes through $R = 1101.0$ and $R = 911.3$.

$$a_1 (1101.0)^2 + b_1 (1101.0) + c_1 = 25.113 \tag{1}$$

$$a_1 (911.3)^2 + b_1 (911.3) + c_1 = 30.131 \tag{2}$$

$a_2 R^2 + b_2 R + c_2$ passes through $R = 911.3$ and $R = 636.0$.

$$a_2 (911.3)^2 + b_2 (911.3) + c_2 = 30.131 \tag{3}$$

$$a_2 (636.0)^2 + b_2 (636.0) + c_2 = 40.120 \tag{4}$$

$a_3 R^2 + b_3 R + c_3$ passes through $R = 636.0$ and $R = 451.1$.

$$a_3 (636.0)^2 + b_3 (636.0) + c_3 = 40.120 \tag{5}$$

$$a_3 (451.1)^2 + b_3 (451.1) + c_3 = 50.128 \tag{6}$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $R = 911.3$

$$2a_1 (911.3) + b_1 - 2a_2 (911.3) - b_2 = 0 \tag{7}$$

At $R = 636.0$

$$2a_2 (636.0) + b_2 - 2a_3 (636.0) - b_3 = 0 \tag{8}$$

3. Assuming the first spline $a_1 R^2 + b_1 R + c_1$ is linear,

$$a_1 = 0 \tag{9}$$

$$\begin{bmatrix} 1.2122 \times 10^6 & 1101.0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8.3047 \times 10^5 & 911.3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.3047 \times 10^5 & 911.3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.0450 \times 10^5 & 636.0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.0450 \times 10^5 & 636.0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0349 \times 10^5 & 451.1 & 1 \\ 1822.6 & 1 & 0 & -1822.6 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1272 & 1 & 0 & -1272 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25.113 \\ 30.131 \\ 30.131 \\ 40.120 \\ 40.120 \\ 50.128 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 9 equations gives the 9 unknowns as

i	a_i	b_i	c_i
1	0	−0.026452	54.237
2	3.5713×10^{-5}	−0.091543	83.895
3	4.3325×10^{-5}	−0.10122	86.974

Therefore, the splines are given by

$$T(R) = -0.026452R + 54.237, \qquad\qquad 911.3 \le R \le 1101.0$$
$$= 3.5713 \times 10^{-5} R^2 - 0.091543R + 83.895, \qquad 636.0 \le R \le 911.3$$
$$= 4.3325 \times 10^{-5} R^2 - 0.10122R + 86.974, \qquad 451.1 \le R \le 636.0$$

At $R = 754.8$

$$T(754.8) = 3.5713 \times 10^{-5}(754.8)^2 - 0.091543(754.8) + 83.895$$
$$= 35.145 \text{ °C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic splines is

$$|\in_a| = \left| \frac{35.145 - 35.809}{35.145} \right| \times 100$$
$$= 1.8892\%$$

b) Since there are four data points, three quadratic splines pass through them.

$$y(x) = a_1 x^2 + b_1 x + c_1, \qquad 7.0040 \le x \le 6.8149$$
$$= a_2 x^2 + b_2 x + c_2, \qquad 6.8149 \le x \le 6.4552$$
$$= a_3 x^2 + b_3 x + c_3, \qquad 6.4552 \le x \le 6.1117$$

The equations are found as follows.
1. Each quadratic spline passes through two consecutive data points.
$a_1 x^2 + b_1 x + c_1$ passes through $x = 7.0040$ and $x = 6.8149$.

$$a_1(7.0040)^2 + b_1(7.0040) + c_1 = 0.039820 \tag{1}$$
$$a_1(6.8149)^2 + b_1(6.8149) + c_1 = 0.033188 \tag{2}$$

$a_2 x^2 + b_2 x + c_2$ passes through $x = 6.8149$ and $x = 6.4552$.

$$a_2(6.8149)^2 + b_2(6.8149) + c_2 = 0.033188 \tag{3}$$
$$a_2(6.4552)^2 + b_2(6.4552) + c_2 = 0.024925 \tag{4}$$

$a_3 x^2 + b_3 x + c_3$ passes through $x = 6.4552$ and $x = 6.1117$.

$$a_3(6.4552)^2 + b_3(6.4552) + c_3 = 0.024925 \tag{5}$$
$$a_3(6.1117)^2 + b_3(6.1117) + c_3 = 0.019949 \tag{6}$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $x = 6.8149$

$$2a_1(6.8149) + b_1 - 2a_2(6.8149) - b_2 = 0 \tag{7}$$

At $x = 6.4552$

$$2a_2(6.4552) + b_2 - 2a_3(6.4552) - b_3 = 0 \tag{8}$$

3. Assuming the first spline $a_1 x^2 + b_1 x + c_1$ is linear,

$$a_1 = 0 \tag{9}$$

$$
\begin{bmatrix}
49.056 & 7.0040 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
46.442 & 6.8149 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 46.442 & 6.8149 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 41.670 & 6.4552 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 41.670 & 6.4552 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 37.353 & 6.1117 & 1 \\
13.630 & 1 & 0 & -13.630 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12.910 & 1 & 0 & -12.910 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3
\end{bmatrix}
=
\begin{bmatrix}
0.039820 \\ 0.033188 \\ 0.033188 \\ 0.024925 \\ 0.024925 \\ 0.019949 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Solving the above 9 equations gives the 9 unknowns as

i	a_i	b_i	c_i
1	0	0.035069	−0.20580
2	0.033627	−0.42326	1.3559
3	−0.010501	0.14646	−0.48289

Therefore, the splines are given by

$$y(x) = 0.035069x - 0.20580, \qquad\qquad 7.0040 \le x \le 6.8149$$
$$= 0.033627x^2 - 0.42326x + 1.3559, \qquad 6.8149 \le x \le 6.4552$$
$$= -0.010501x^2 + 0.14646x - 0.48289, \qquad 6.4552 \le x \le 6.1117$$

At $x = \ln(754.8)$

$$y(\ln(754.8)) = 0.033627(\ln(754.8))^2 - 0.42326(\ln(754.8)) + 1.3559$$
$$= 0.027775$$

Since $T = \dfrac{1}{y}$,

$$T = \frac{1}{0.027800}$$
$$= 36.004\,°C$$

Since the actual measured value at 754.8 ohms is 35.285 °C, the absolute relative true error between the value used for part (a) is

$$\left|\in_t\right| = \left|\frac{35.285 - 35.145}{35.285}\right| \times 100$$

$$= 0.39543\%$$

and for part (b) is

$$\left|\in_t\right| = \left|\frac{35.285 - 36.004}{35.285}\right| \times 100$$

$$= 2.0381\%$$

Therefore, the spline method of interpolation using quadratic splines, that is,

$$T(R) = a_0 + a_1 R + a_2 R^2$$

obtained more accurate results than the calibration curve of

$$\frac{1}{T} = a_0 + a_1 [\ln R] + a_2 [\ln R]^2$$

INTERPOLATION	
Topic	Spline Method of Interpolation
Summary	Textbook notes on the spline method of interpolation
Major	Electrical Engineering
Authors	Autar Kaw, Michael Keteltas
Date	November 12, 2012
Web Site	http://numericalmethods.eng.usf.edu