Romberg Rule of Integration

Electrical Engineering Majors

Authors: Autar Kaw, Charlie Barker

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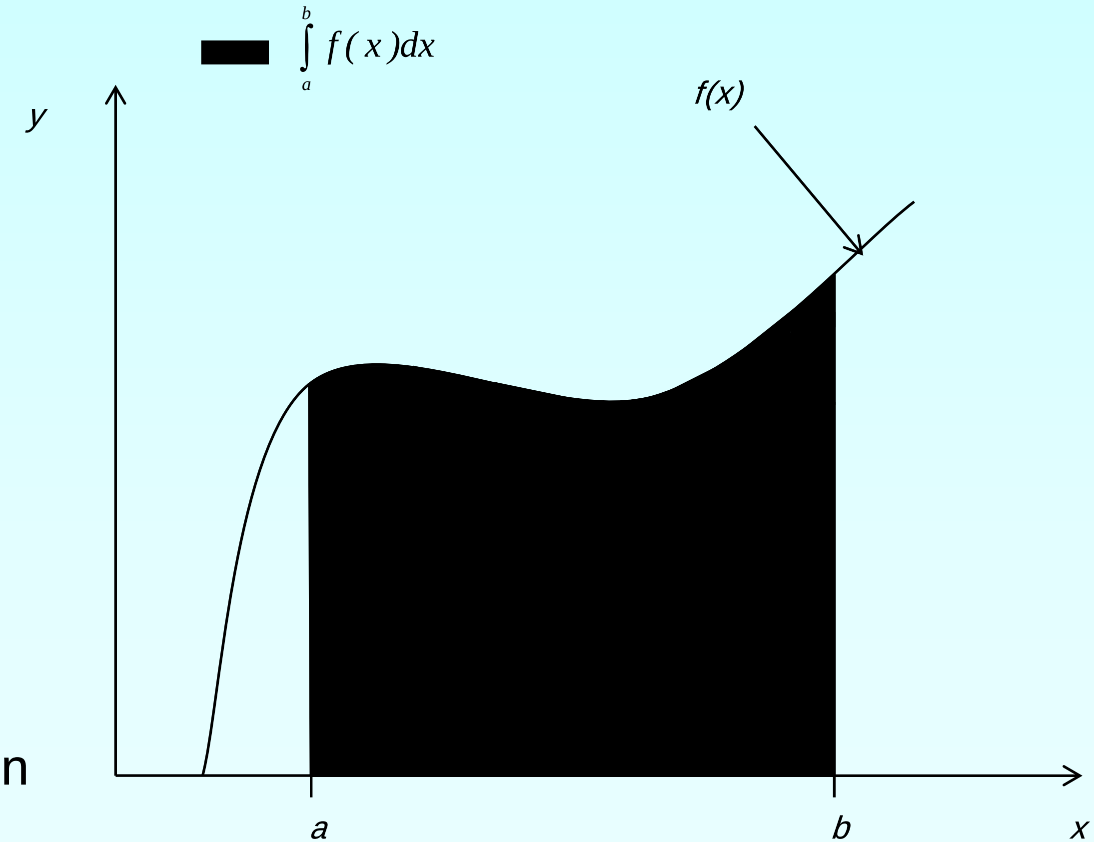
Basis of Romberg Rule

Integration

The process of measuring the area under a curve.

$$I = \int_a^b f(x)dx$$



Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration

What is The Romberg Rule?

Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Error in Multiple Segment Trapezoidal Rule

The true error in a multiple segment Trapezoidal Rule with n segments for an integral

$$I=\int_a^b f(x)dx$$

Is given by

$$E_t=\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

where for each *i*, ξ_i is a point somewhere in the domain , $[a+(i-1)h, a+ih]$.

Error in Multiple Segment Trapezoidal Rule

The term $\dfrac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$.

This leads us to say that the true error, E_t previously defined can be approximated as

$$E_t \cong \alpha \frac{1}{n^2}$$

Error in Multiple Segment Trapezoidal Rule

Table 1 shows the results obtained for the integral using multiple segment Trapezoidal rule for

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$\left\|\in_t\right\|\%$	$\left\|\in_a\right\|\%$
1	11868	807	7.296	---
2	11266	205	1.854	5.343
3	11153	91.4	0.8265	1.019
4	11113	51.5	0.4655	0.3594
5	11094	33.0	0.2981	0.1669
6	11084	22.9	0.2070	0.09082
7	11078	16.8	0.1521	0.05482
8	11074	12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Error in Multiple Segment Trapezoidal Rule

The true error gets approximately quartered as the number of segments is doubled. This information is used to get a better approximation of the integral, and is the basis of Richardson's extrapolation.

Richardson's Extrapolation for Trapezoidal Rule

The true error, E_t in the *n*-segment Trapezoidal rule is estimated as

$$E_t \approx \frac{C}{n^2}$$

where *C* is an *approximate constant* of proportionality. Since

$$E_t = TV - I_n$$

Where TV = true value and I_n = approx. value

Richardson's Extrapolation for Trapezoidal Rule

From the previous development, it can be shown that

$$\frac{C}{(2n)^2} \approx TV - I_{2n}$$

when the segment size is doubled and that

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

which is Richardson's Extrapolation.

Example 1

The probability for an oscillator to have its frequency within 5% of the target of 1kHz is determined by finding total area under the normal distribution function for the range in question:

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

a) Use Richardson's rule to find the distance covered. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) $I_2 = 0.99637$

$I_4 = 0.96969$

Table. Values obtained for Trapezoidal Rule

n	Trapezoidal Rule
1	0.11489
2	0.99637
4	0.96969
8	0.97901

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing *n=2*,

$$TV \approx I_4 + \frac{I_4 - I_2}{3} = 0.96969 + \frac{0.96969 - 0.99637}{3} = 0.96078$$

Solution (cont.)

b) Since the exact value of the above integral cannot be found, we take numerical integration value using maple as exact value

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=0.98236$$

Hence

$$E_t = True\ Value - Approximate\ Value$$
$$= 0.98236 - 0.96078$$
$$= -0.021560$$

Solution (cont.)

c) The absolute relative true error $\left|\in_t\right|$ would then be

$$\left|\in_t\right| = \left|\frac{0.98236 - 0.96078}{0.98236}\right| \times 100$$

$$= 2.1947\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Solution (cont.)

Table 2 The values obtained using Richardson's extrapolation formula for Trapezoidal rule for

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

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n	**Trapezoidal Rule**	$\lvert\in_t\rvert$ **for Trapezoidal Rule**	**Richardson's Extrapolation**	$\lvert\in_t\rvert$ **for Richardson's Extrapolation**
1	0.11489	88.3	--	--
2	0.99637	1.427	1.2902	31.337
4	0.96969	1.289	0.96078	2.1947
8	0.97901	0.3404	0.98212	0.024422

Romberg Integration

Romberg integration is same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg Integration

Note that the variable *TV* is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by = sign.

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Where Ch^4 is an approximation of the true error.

Romberg Integration

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

It follows from the two previous expressions that the true value TV can be written as

$$TV \approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

Romberg Integration

A general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index *k* represents the order of extrapolation. k=1 represents the values obtained from the regular Trapezoidal rule, k=2 represents values obtained using the true estimate as $O(h^2)$. The index *j* represents the more and less accurate estimate of the integral.

Example 2

The probability for an oscillator to have its frequency within 5% of the target of 1kHz is determined by finding total area under the normal distribution function for the range in question:

$$(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given in the Table 1.

Solution

From the table, the needed values from original Trapezoidal rule are

$$I_{1,1} = 0.11489 \qquad I_{1,2} = 0.99637$$

$$I_{1,3} = 0.96969 \qquad I_{1,4} = 0.97901$$

Table. Values obtained for Trapezoidal Rule

n	Trapezoidal Rule
1	0.11489
2	0.99637
4	0.96969
8	0.97901

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively.

Solution (cont.)

To get the first order extrapolation values,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= 0.99637 + \frac{0.99637 - 0.11489}{3}$$

$$= 1.2902$$

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= 0.96969 + \frac{0.96969 - 0.99637}{3}$$

$$= 0.96080$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= 0.97901 + \frac{0.97901 - 0.96969}{3}$$

$$= 0.98212$$

Solution (cont.)

For the second order extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 0.96079 + \frac{0.96080 - 1.2902}{15}$$

$$= 0.93884$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 0.98212 + \frac{0.98212 - 0.96080}{15}$$

$$= 0.98354$$

Solution (cont.)

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 0.98354 + \frac{0.98354 - 0.93884}{63}$$

$$= 0.98425$$

Table 3 shows these increased correct values in a tree graph.

Solution (cont.)

Table 3: Improved estimates of the integral value using Romberg Integration

		1st Order	2nd Order	3rd Order
1-segment	0.11489			
		1.2902		
			0.93884	
2-segment	0.99637			
				0.98425
		0.96080		
			0.98354	
4-segment	0.96969			
		0.98212		
8-segment	0.97901			

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/romberg_method.html

THE END