# Chapter 08.04
# Runge-Kutta 4th Order Method for Ordinary Differential Equations

*After reading this chapter, you should be able to*
1. *develop Runge-Kutta 4th order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4th order method*

**What is the Runge-Kutta 4th order method?**

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

**How does one write a first order differential equation in the above form?**

**Example 1**

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \ \text{form.}$$

**Solution**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \ y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \ y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

**Example 2**

Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \ \text{form.}$$

**Solution**

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y}, \ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)h \tag{1}$$

where knowing the value of $y = y_i$ at $x_i$, we can find the value of $y = y_{i+1}$ at $x_{i+1}$, and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i,y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2 y}{dx^2}\Big|_{x_i,y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3 y}{dx^3}\Big|_{x_i,y_i}(x_{i+1} - x_i)^3$$

$$+ \frac{1}{4!}\frac{d^4 y}{dx^4}\Big|_{x_i,y_i}(x_{i+1} - x_i)^4 \tag{2}$$

Knowing that $\dfrac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4 \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{4}$$

$$k_1 = f(x_i, y_i) \tag{5a}$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \tag{5b}$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \tag{5c}$$

$$k_4 = f(x_i + h, y_i + k_3 h) \tag{5d}$$

**Example 3**

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of $150\,\mu F$, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(t))| - 2 - v(t)}{0.04}, 0\right)\right\}$$

$$v(0) = 0$$

Using the Runge-Kutta $4^{\text{th}}$ order method, find voltage across the capacitor at $t = 0.00004\,\text{s}$. Use step size $h = 0.00002\,\text{s}$.

**Solution**

$$\frac{dv}{dt} = \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(t))| - 2 - v}{0.04}, 0\right)\right\}$$

$$f(t,v) = \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(t))| - 2 - v}{0.04}, 0\right)\right\}$$

$$v_{i+1} = v_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $v_0 = 0$

$$k_1 = f(t_0, v_0)$$

$$= f(0,0)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0))| - 2 - (0)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(400,0)\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + 400\}$$

$$= 2.6660 \times 10^{6}$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_1h\right)$$

$$= f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(2.6660 \times 10^6)0.00002\right)$$

$$= f(0.00001, 26.660)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (26.660)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-266.50, 0)\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 - 266.50\}$$

$$= -666.67$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_2h\right)$$

$$= f\left(0 + \frac{1}{2}(0.00002), 0 + \frac{1}{2}(-666.67)0.00002\right)$$

$$= f(0.00001, -0.0066667)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00001))| - 2 - (-0.0066667)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(400.16, 0)\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + 400.16\}$$

$$= 2.6671 \times 10^6$$

$$k_4 = f(t_0 + h, v_0 + k_3h)$$

$$= f(0 + 0.00002, 0 + (2.6671 \times 10^6)0.00002)$$

$$= f(0.00002, 53.342)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{|18\cos(120\pi(0.00002))| - 2 - (53.342)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + \max(-933.56, 0)\}$$

$$= \frac{1}{150 \times 10^{-6}}\{-0.1 + 0\}$$

$$= -666.67$$

$$v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 0 + \frac{1}{6}(2.6660 \times 10^6 + 2(-666.67) + 2(2.6671 \times 10^6) + (-666.67))0.00002$$

$$= 0 + \frac{1}{6}\left(7.9982 \times 10^6\right)0.00002$$

$$= 26.661 \text{ V}$$

$v_1$ is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$

$$v(0.00002) \approx v_1 = 26.661 \text{V}$$

For $i = 1, t_1 = 0.00002, v_1 = 26.661$

$$k_1 = f(t_1, v_1)$$

$$= f(0.00002, 26.661)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00002))\right| - 2 - (26.661)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max(-266.51, 0)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + 0\right\}$$

$$= -666.67$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)$$

$$= f(0.00003, 26.654)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.654)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max(-266.35, 0)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + 0\right\}$$

$$= -666.67$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_2 h\right)$$

$$= f\left(0.00002 + \frac{1}{2}(0.00002), 26.661 + \frac{1}{2}(-666.67)0.00002\right)$$

$$= f(0.00003, 26.654)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.654)}{0.04}, 0\right)\right\}$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max(-266.35, 0)\right\}$$

$$= \frac{1}{150 \times 10^{-6}} \{-0.1 + 0\}$$

$$= -666.67$$

$$k_4 = f(t_1 + h, v_1 + k_3 h)$$

$$= f(0.00002 + (0.00002), 26.661 + (-666.67)0.00002)$$

$$= f(0.00003, 26.647)$$

$$= \frac{1}{150 \times 10^{-6}} \left\{ -0.1 + \max\left( \frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.634)}{0.04}, 0 \right) \right\}$$

$$= \frac{1}{150 \times 10^{-6}} \{-0.1 + \max(-265.87, 0)\}$$

$$= \frac{1}{150 \times 10^{-6}} \{-0.1 + 0\}$$

$$= -666.67$$

$$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 26.661 + \frac{1}{6}(-666.67 + 2(-666.67) + 2(-666.67) + (-666.67))0.00002$$

$$= 26.661 + \frac{1}{6}(-4000.0)0.00002$$

$$= 26.647 \text{ V}$$

$v_2$ is the approximate voltage at $t = t_2$

$$t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004 \text{ s}$$

$$v(.00004) \approx v_2 = 26.647 \text{ V}$$

Figure 1 compares the exact solution of $v(0.00004) = 15.974$ V with the numerical solution using Runge-Kutta 4$^{th}$ order method step size of $h = 0.00002$ s .
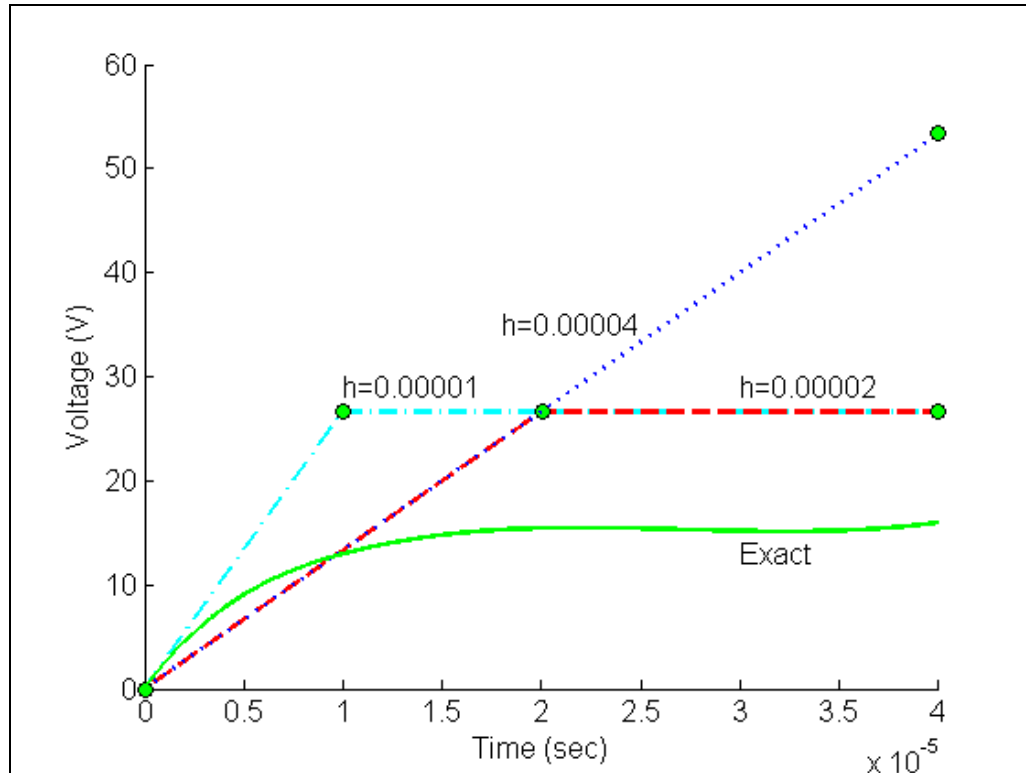
**Figure 1** Comparison of Runge-Kutta 4$^{th}$ order method with exact solution   for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated temperature at $t = 0.00004\,\text{s}$.

**Table 1**  Value of voltage at time, $t = 0.00004\,\text{s}$  for different step sizes.

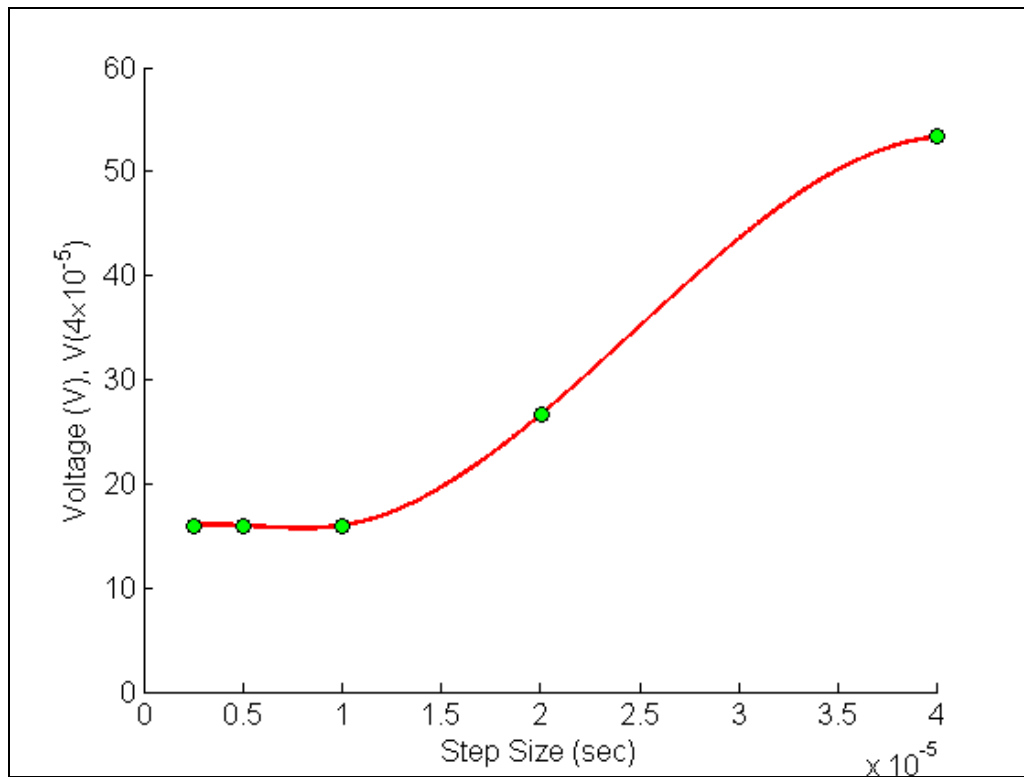| Step size, $h$ | $v(0.00004)$ | $E_t$ | $\lvert \in_t \rvert\,\%$ |
|---|---|---|---|
| 0.00004 | 53.335 | −37.361 | 233.89 |
| 0.00002 | 26.647 | −10.673 | 66.817 |
| 0.00001 | 15.986 | −0.012299 | 0.076996 |
| 0.000005 | 15.975 | −0.00050402 | 0.0031552 |
| 0.0000025 | 15.976 | −0.0015916 | 0.0099639 |

**Figure 2** Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.
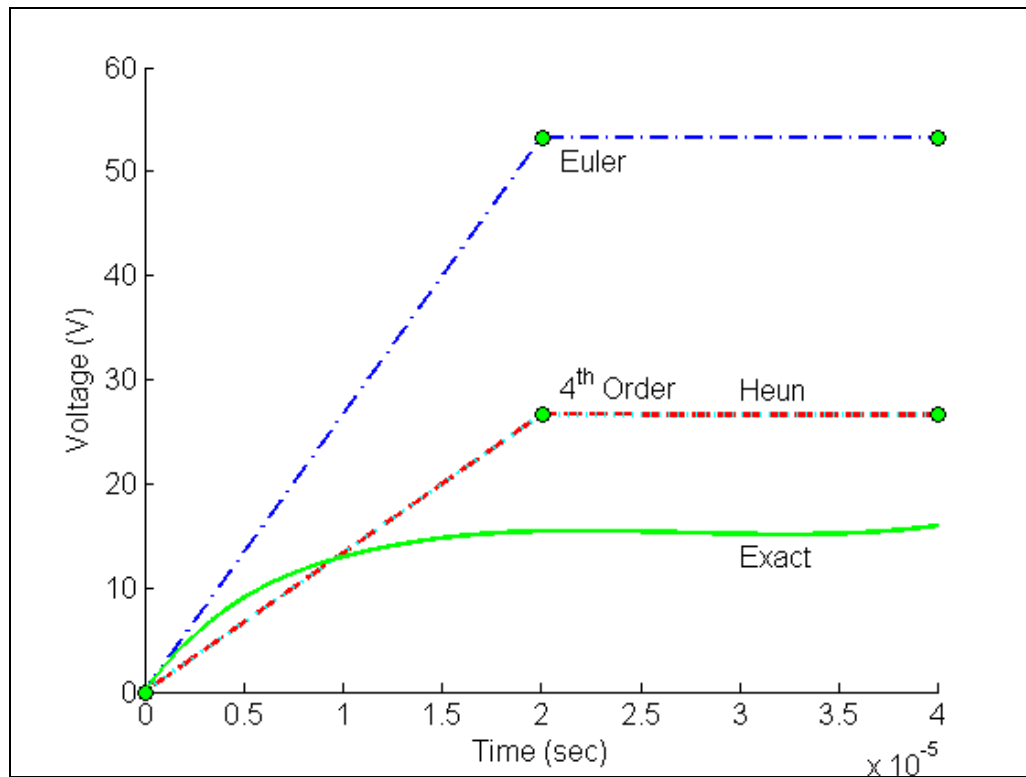
**Figure 3**  Comparison of Runge-Kutta methods of 1$^{st}$, 2$^{nd}$, and 4$^{th}$ order.

| ORDINARY DIFFERENTIAL EQUATIONS | |
|---|---|
| Topic | Runge-Kutta 4th order method |
| Summary | Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations. |
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| Authors | Autar Kaw |
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| Web Site | http://numericalmethods.eng.usf.edu |