Measuring Errors

Major: All Engineering Majors

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http://numericalmethods.eng.usf.edu
Numerical Methods for STEM undergraduates
Why measure errors?

1) To determine the accuracy of numerical results.

2) To develop stopping criteria for iterative algorithms.
True Error

- Defined as the difference between the true value in a calculation and the approximate value found using a numerical method etc.

True Error = True Value - Approximate Value
Example—True Error

The derivative, \( f'(x) \) of a function \( f(x) \) can be approximated by the equation,

\[
f''(x) \approx \frac{f(x + h) - f(x)}{h}
\]

If \( f(x) = 7e^{0.5x} \) and \( h = 0.3 \)

a) Find the approximate value of \( f'(2) \)

b) True value of \( f'(2) \)

c) True error for part (a)
Solution:

a) For $x = 2$ and $h = 0.3$

$$f'(2) \approx \frac{f(2 + 0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3} = 10.263$$
Example (cont.)

Solution:

b) The exact value of $f''(2)$ can be found by using our knowledge of differential calculus.

\[ f(x) = 7e^{0.5x} \]
\[ f'(x) = 7 \times 0.5 \times e^{0.5x} \]
\[ = 3.5e^{0.5x} \]

So the true value of $f''(2)$ is

\[ f''(2) = 3.5e^{0.5(2)} \]
\[ = 9.5140 \]

True error is calculated as

\[ E_t = \text{True Value} - \text{Approximate Value} \]
\[ = 9.5140 - 10.263 = -0.722 \]
Relative True Error

- Defined as the ratio between the true error, and the true value.

\[
\text{Relative True Error } (\varepsilon_i) = \frac{\text{True Error}}{\text{True Value}}
\]
Example—Relative True Error

Following from the previous example for true error, find the relative true error for \( f(x) = 7e^{0.5x} \) at \( f'(2) \) with \( h = 0.3 \)

From the previous example,

\[ E_t = -0.722 \]

Relative True Error is defined as

\[ \varepsilon_t = \frac{\text{True Error}}{\text{True Value}} \]

\[ = \frac{-0.722}{9.5140} = -0.075888 \]

as a percentage,

\[ \varepsilon_t = -0.075888 \times 100\% = -7.5888\% \]
Approximate Error

- What can be done if true values are not known or are very difficult to obtain?
- Approximate error is defined as the difference between the present approximation and the previous approximation.

Approximate Error ($E_a$) = Present Approximation - Previous Approximation
Example—Approximate Error

For \( f(x) = 7e^{0.5x} \) at \( x = 2 \) find the following,

a) \( f'(2) \) using \( h = 0.3 \)

b) \( f'(2) \) using \( h = 0.15 \)

c) approximate error for the value of \( f'(2) \) for part b)

Solution:

a) For \( x = 2 \) and \( h = 0.3 \)

\[
f'(x) \approx \frac{f(x + h) - f(x)}{h}
\]

\[
f'(2) \approx \frac{f(2 + 0.3) - f(2)}{0.3}
\]
Example (cont.)

Solution: (cont.)

\[
\frac{f(2.3) - f(2)}{0.3} = \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}
\]

\[
= \frac{22.107 - 19.028}{0.3} = 10.263
\]

b) For \( x = 2 \) and \( h = 0.15 \)

\[
f'(2) \approx \frac{f(2 + 0.15) - f(2)}{0.15}
\]

\[
= \frac{f(2.15) - f(2)}{0.15}
\]
Example (cont.)

Solution: (cont.)

\[ \frac{7e^{0.5(2.15)} - 7e^{0.5(2)}}{0.15} \]
\[ = \frac{20.50 - 19.028}{0.15} = 9.8800 \]

c) So the approximate error, \( E_a \) is

\[ E_a = \text{Present Approximation} - \text{Previous Approximation} \]
\[ = 9.8800 - 10.263 \]
\[ = -0.38300 \]
Relative Approximate Error

- Defined as the ratio between the approximate error and the present approximation.

Relative Approximate Error ($\varepsilon_a$) = \[
\frac{\text{Approximate Error}}{\text{Present Approximation}}
\]
Example—Relative Approximate Error

For \( f(x) = 7e^{0.5x} \) at \( x = 2 \), find the relative approximate error using values from \( h = 0.3 \) and \( h = 0.15 \)

Solution:
From Example 3, the approximate value of \( f'(2) = 10.263 \) using \( h = 0.3 \) and \( f'(2) = 9.8800 \) using \( h = 0.15 \)

\[
E_a = \text{Present Approximation} - \text{Previous Approximation}
\]
\[
= 9.8800 - 10.263
\]
\[
= -0.38300
\]
Example (cont.)

Solution: (cont.)

\[
\varepsilon_a = \frac{\text{Approximate Error}}{\text{Present Approximation}}
\]

\[
= \frac{-0.38300}{9.8800} = -0.038765
\]

as a percentage,

\[
\varepsilon_a = -0.038765 \times 100\% = -3.8765\%
\]

Absolute relative approximate errors may also need to be calculated,

\[
|\varepsilon_a| = |-0.038765| = 0.038765 \text{ or } 3.8765\%
\]
How is Absolute Relative Error used as a stopping criterion?

If \(|\varepsilon_a| \leq \varepsilon_s\) where \(\varepsilon_s\) is a pre-specified tolerance, then no further iterations are necessary and the process is stopped.

If at least \(m\) significant digits are required to be correct in the final answer, then

\[ |\varepsilon_a| \leq 0.5 \times 10^{2-m} \]
Table of Values

For $f(x) = 7e^{0.5x}$ at $x = 2$ with varying step size, $h$

| $h$  | $f''(2)$  | $|\varepsilon_a|$ | $m$   |
|------|-----------|------------------|------|
| 0.3  | 10.263    | N/A              | 0    |
| 0.15 | 9.8800    | 0.038765%        | 3    |
| 0.10 | 9.7558    | 0.012731%        | 3    |
| 0.01 | 9.5378    | 0.024953%        | 3    |
| 0.001| 9.5164    | 0.002248%        | 4    |