Sources of Error

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http://numericalmethods.eng.usf.edu

Numerical Methods for STEM undergraduates

Two sources of numerical error

1) Round off error
2) Truncation error

Round off Error

- Caused by representing a number approximately

$$\frac{1}{3} \cong 0.333333$$

$$\sqrt{2} \cong 1.4142...$$

Problems created by round off error

- 28 Americans were killed on February 25, 1991 by an Iraqi Scud missile in Dhahran, Saudi Arabia.
- The patriot defense system failed to track and intercept the Scud. Why?

Problem with Patriot missile

- Clock cycle of 1/10 seconds was represented in 24-bit fixed point register created an error of 9.5×10^{-8} seconds.
- The battery was on for 100 consecutive hours, thus causing an inaccuracy of

$$= 9.5 \times 10^{-8} \frac{\text{s}}{0.1\text{s}} \times 100\text{hr} \times \frac{3600\text{s}}{1\text{hr}}$$
$$= 0.342s$$

Problem (cont.)

- The shift calculated in the ranging system of the missile was 687 meters.
- The target was considered to be out of range at a distance greater than 137 meters.

Truncation error

- Error caused by truncating or approximating a mathematical procedure.

Example of Truncation Error

Taking only a few terms of a Maclaurin series to approximate e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

If only 3 terms are used,

$$Truncation\ \ Error = e^x - \left(1 + x + \frac{x^2}{2!}\right)$$

Another Example of Truncation Error

Using a finite Δx to approximate $f'(x)$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

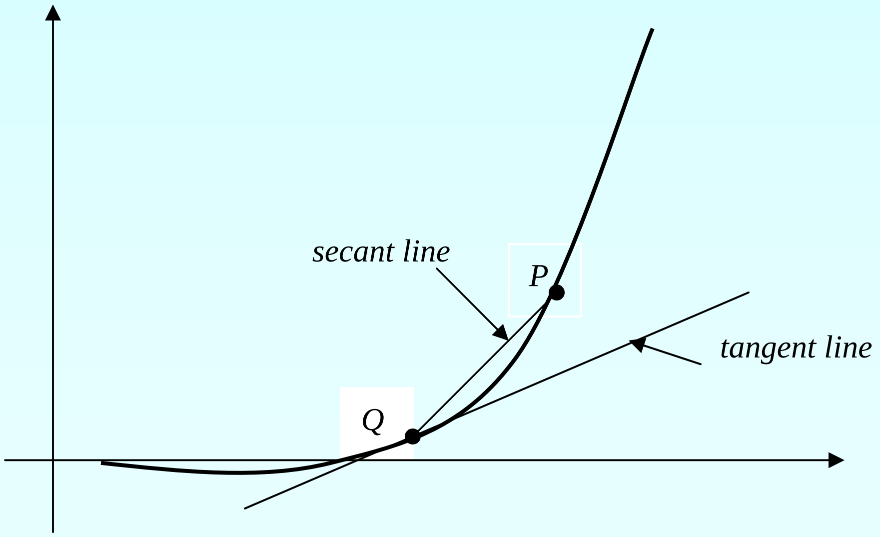
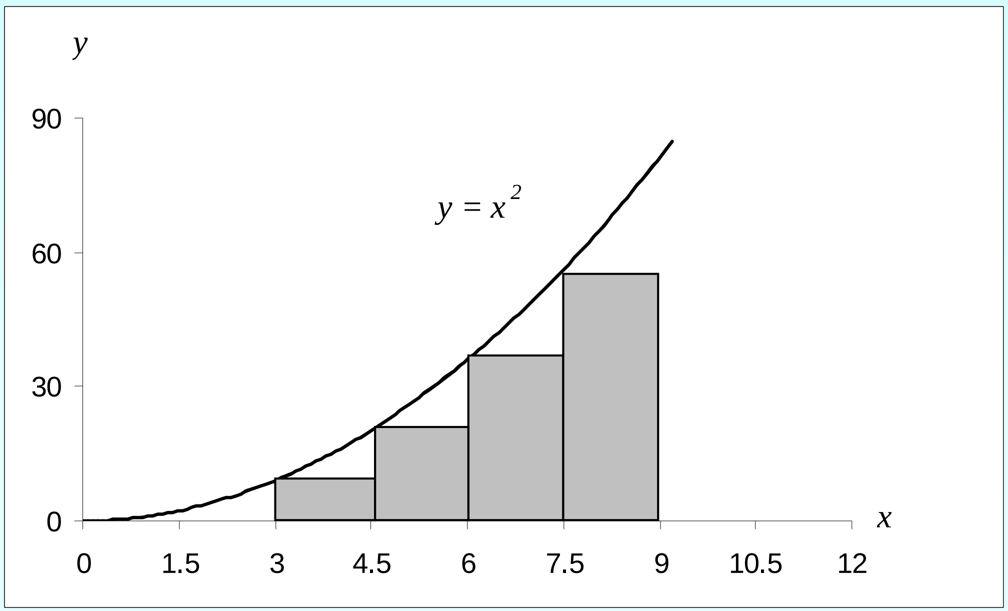


Figure 1. Approximate derivative using finite Δx

Another Example of Truncation Error

Using finite rectangles to approximate an integral.



Example 1 —Maclaurin series

Calculate the value of $e^{1.2}$ with an absolute relative approximate error of less than 1%.

$$e^{1.2} = 1 + 1.2 + \frac{1.2^2}{2!} + \frac{1.2^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

n	$e^{1.2}$	E_a	$\left\|\in_a\right\|\%$
1	1	—	—
2	2.2	1.2	54.545
3	2.92	0.72	24.658
4	3.208	0.288	8.9776
5	3.2944	0.0864	2.6226
6	3.3151	0.020736	0.62550

6 terms are required

Example 2 —Differentiation

Find $f'(3)$ for $f(x)=x^2$ using $f'(x)\approx\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

and $\Delta x = 0.2$

$$f^{'}(3)=\frac{f(3+0.2)-f(3)}{0.2}$$

$$=\frac{f(3.2)-f(3)}{0.2}$$

$$=\frac{3.2^2-3^2}{0.2}=\frac{10.24-9}{0.2}=\frac{1.24}{0.2}=6.2$$

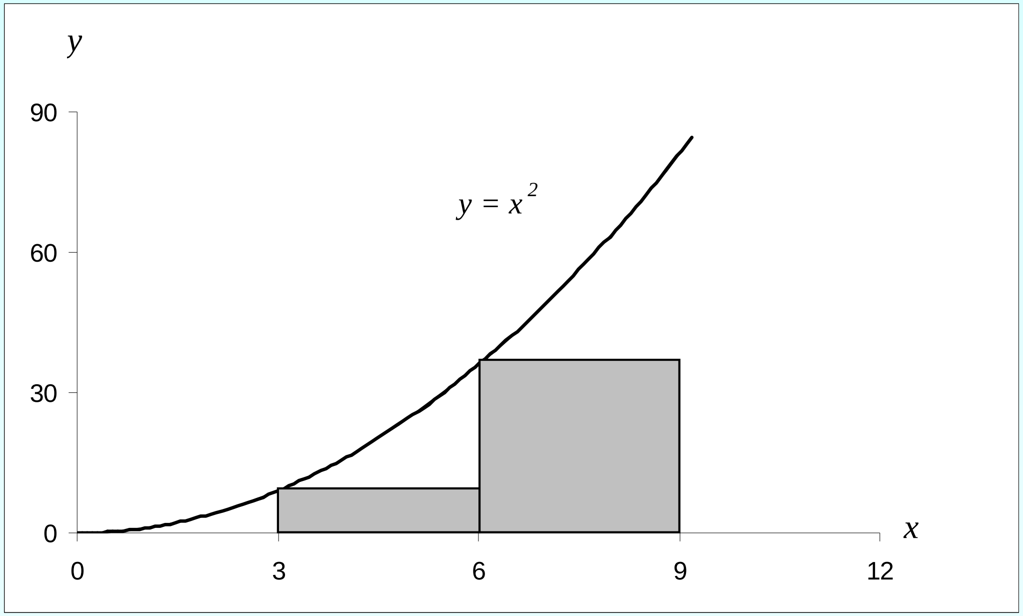
The actual value is

$$f^{'}(x)=2x,\quad f^{'}(3)=2\times 3=6$$

Truncation error is then, $6-6.2=-0.2$

Example 3 — Integration

Use two rectangles of equal width to approximate the area under the curve for $f(x) = x^2$ over the interval $[3,9]$



$$\int_3^9 x^2 dx$$

Integration example (cont.)

Choosing a width of 3, we have

$$\int_3^9 x^2 dx = (x^2)\Big|_{x=3}(6-3) + (x^2)\Big|_{x=6}(9-6)$$

$$= (3^2)3 + (6^2)3$$

$$= 27 + 108 = 135$$

Actual value is given by

$$\int_3^9 x^2 dx = \left[\frac{x^3}{3}\right]_3^9 = \left[\frac{9^3 - 3^3}{3}\right] = 234$$

Truncation error is then

$$234 - 135 = 99$$