Taylor Series Revisited

Major: All Engineering Majors

Authors: Autar Kaw, Luke Snyder

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Numerical Methods for STEM undergraduates
What is a Taylor series?

Some examples of Taylor series which you must have seen

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]
General Taylor Series

The general form of the Taylor series is given by

\[ f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \ldots \]

provided that all derivatives of \( f(x) \) are continuous and exist in the interval \([x, x+h]\)

What does this mean in plain English?

As Archimedes would have said, “Give me the value of the function at a single point, and the value of all (first, second, and so on) its derivatives at that single point, and I can give you the value of the function at any other point” (fine print excluded)
Example—Taylor Series

Find the value of \( f(6) \) given that \( f(4) = 125, \ f'(4) = 74, \ f''(4) = 30, \ f'''(4) = 6 \) and all other higher order derivatives of \( f(x) \) at \( x = 4 \) are zero.

Solution:

\[
f(x + h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots
\]

\[
x = 4
\]

\[
h = 6 - 4 = 2
\]
Example (cont.)

Solution: (cont.)

Since the higher order derivatives are zero,

\[ f(4 + 2) = f(4) + f''(4)2 + f'''(4)\frac{2^2}{2!} + f''''(4)\frac{2^3}{3!} \]

\[ f(6) = 125 + 74(2) + 30\left(\frac{2^2}{2!}\right) + 6\left(\frac{2^3}{3!}\right) \]

\[ = 125 + 148 + 60 + 8 \]

\[ = 341 \]

Note that to find \( f(6) \) exactly, we only need the value of the function and all its derivatives at some other point, in this case \( x = 4 \).
Derivation for Maclaurin Series for $e^x$

Derive the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Maclaurin series is simply the Taylor series about the point $x=0$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \frac{f''''(x)h^4}{4} + \frac{f'''''(x)h^5}{5} + \cdots$$

$$f(0+h) = f(0) + f'(0)h + \frac{f''(0)h^2}{2!} + \frac{f'''(0)h^3}{3!} + \frac{f''''(0)h^4}{4} + \frac{f'''''(0)h^5}{5} + \cdots$$
Derivation (cont.)

Since \( f(x) = e^x \), \( f'(x) = e^x \), \( f''(x) = e^x \), ..., \( f^{(n)}(x) = e^x \) and 
\( f^{(n)}(0) = e^0 = 1 \)

the Maclaurin series is then

\[
f(h) = (e^0) + (e^0)h + \frac{(e^0)}{2!} h^2 + \frac{(e^0)}{3!} h^3 \ldots
\]

\[= 1 + h + \frac{1}{2!} h^2 + \frac{1}{3!} h^3 \ldots\]

So,

\[
f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]
Error in Taylor Series

The Taylor polynomial of order $n$ of a function $f(x)$ with $(n+1)$ continuous derivatives in the domain $[x, x+h]$ is given by

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \cdots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x)$$

where the remainder is given by

$$R_n(x) = \frac{(x-h)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

where

$$x < c < x+h$$

that is, $c$ is some point in the domain $[x, x+h]$
Example—error in Taylor series

The Taylor series for $e^x$ at point $x = 0$ is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

It can be seen that as the number of terms used increases, the error bound decreases and hence a better estimate of the function can be found.

How many terms would it require to get an approximation of $e^1$ within a magnitude of true error of less than $10^{-6}$.
Example—(cont.)

Solution:

Using \((n+1)\) terms of Taylor series gives error bound of

\[
R_n(x) = \frac{(x-h)^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad x = 0, h = 1, f(x) = e^x
\]

\[
R_n(0) = \frac{(0-1)^{n+1}}{(n+1)!} f^{(n+1)}(c)
\]

\[
= \frac{(-1)^{n+1}}{(n+1)!} e^c
\]

Since

\[
x < c < x + h
\]

\[
0 < c < 0 + 1
\]

\[
0 < c < 1
\]

\[
\frac{1}{(n+1)!} < |R_n(0)| < \frac{e}{(n+1)!}
\]
Example—(cont.)

Solution: (cont.)

So if we want to find out how many terms it would require to get an approximation of $e^1$ within a magnitude of true error of less than $10^{-6}$,

\[
\frac{e}{(n+1)!} < 10^{-6}
\]

\[(n+1)! > 10^6 e\]

\[(n+1)! > 10^6 \times 3\]

\[n \geq 9\]

So 9 terms or more are needed to get a true error less than $10^{-6}$.