Differentiation-Discrete Functions

Industrial Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Differentiation –Discrete Functions

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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

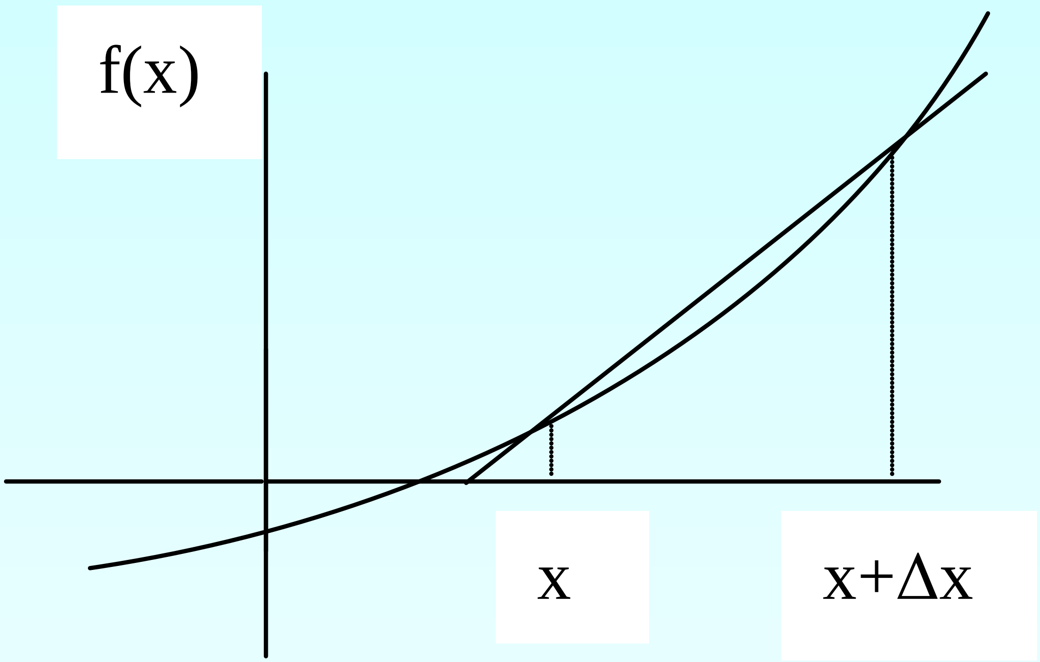

Figure 1 Graphical Representation of forward difference approximation of first derivative.

Example 1

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = \frac{R'(t)}{R(t)}$$

Where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 1.

Table 1 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Using the forward divided difference method, find the failure rate of the DMFC system at $t = 50$ hours.

Example 1 Cont.

Solution

$$R'(t_i) \approx \frac{R(t_{i+1}) - R(t_i)}{\Delta t}$$

$$t_i = 10$$

$$t_{i+1} = 100$$

$$\Delta t = t_{i+1} - t_i$$
$$= 100 - 10$$
$$= 90$$

$$R'(50) \approx \frac{R(100) - R(10)}{90}$$
$$\approx \frac{0.9980 - 0.9998}{90}$$
$$\approx -2.0000 \times 10^{-5}$$

Example 1 Cont.

The reliability $R(t)$ at hours $t=50$ is

$$R(50)\approx\frac{R(100)-R(10)}{100-10}(50-10)+R(10)$$

$$\approx\left(-2.0000\times10^{-5}\right)(40)+0.9998$$

$$\approx 0.999$$

The failure rate $h(t)$ at $t=50$ hours is then

$$h(50)=-\frac{R'(50)}{R(50)}$$

$$\approx-\left(\frac{-2.0000\times10^{-5}}{0.999}\right)$$

$$\approx 2.0020\times10^{-5}$$

Direct Fit Polynomials

In this method, given $'n+1'$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1} x^{n-2} + n a_n x^{n-1}$$

Similarly other derivatives can be found.

Example 2-Direct Fit Polynomials

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = \frac{R'(t)}{R(t)}$$

Where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 2.

Table 2 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Using a third order polynomial interpolant for reliability $R(t)$, find the failure rate of the DMFC system at $t = 50$ hours.

Example 2-Direct Fit Polynomials cont.

Solution

For the third order polynomial (also called cubic interpolation), we choose the reliability given by

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the reliability at $t = 50$, and we are using third order polynomial, we need to choose the four points closest to $t = 50$ and that also bracket $t = 50$ to evaluate it.

The four points are $t_0 = 1$, $t_1 = 10$, $t_2 = 100$ and $t_3 = 1000$ hours.

$$t_o = 1, \; R(t_o) = 0.9999$$

$$t_1 = 10, \; R(t_1) = 0.9998$$

$$t_2 = 100, \; R(t_2) = 0.9980$$

$$t_3 = 1000, \; R(t_3) = 0.9802$$

Example 2-Direct Fit Polynomials cont.

such that

$$R(1) = 0.9999 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$

$$R(10) = 0.9998 = a_0 + a_1(10) + a_2(1)0^2 + a_3(10)^3$$

$$R(100) = 0.9980 = a_0 + a_1(100) + a_2(100)^2 + a_3(100)^3$$

$$R(1000) = 0.9802 = a_0 + a_1(1000) + a_2(1000)^2 + a_3(1000)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 10000 & 1\times10^6 \\ 1 & 1000 & 1\times10^6 & 1\times10^9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.9999 \\ 0.9998 \\ 0.9980 \\ 0.9802 \end{bmatrix}$$

Example 2-Direct Fit Polynomials cont.

Solving the above four equations gives

$$a_0 = 0.9991$$

$$a_1 = -1.0023\times10^{-5}$$

$$a_2 = -9.9788\times10^{-8}$$

$$a_3 = 9.0101\times10^{-11}$$

Hence

$$\begin{aligned} R(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ &= 0.99991 - 1.0023\times10^{-5} t - 9.9788\times10^{-8} t^2 + 9.0101\times10^{-11} t^3, \quad 1 \le t \le 1000 \end{aligned}$$

Example 2-Direct Fit Polynomials cont.

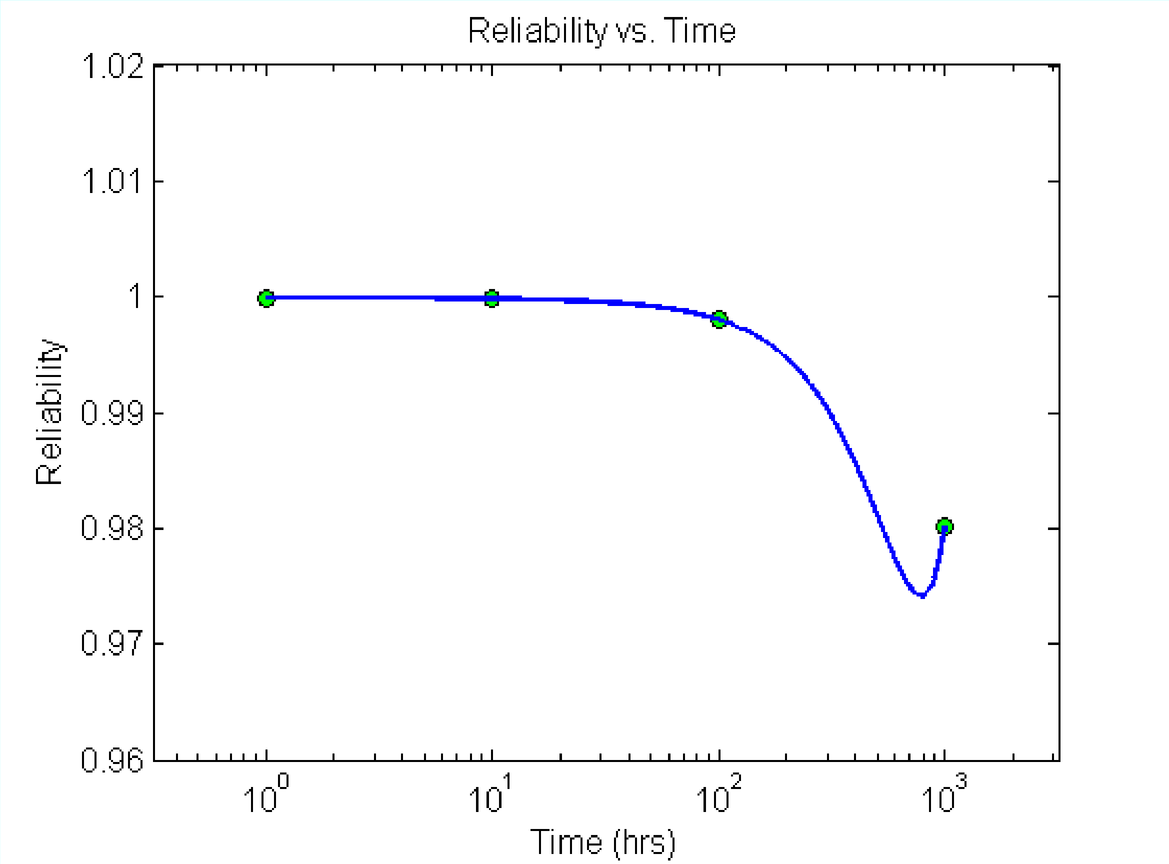

Figure 2 Graph of reliability as a function of time.

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Example 2-Direct Fit Polynomials cont.

The reliability at $t = 50$ is given by,

$$R'(50) = \frac{d}{dt} R(t)\Big|_{t=50}$$

Given that

$$R(t) = 0.99991 - 1.0023\times10^{-5} t - 9.9788\times10^{-8} t^2 + 9.0101\times10^{-11} t^3, \quad 1 \le t \le 1000$$

$$\begin{aligned}
R'(t) &= \frac{d}{dt} R(t) \\
&= \frac{d}{dt}\left(0.99991 - 1.0023\times10^{-5} t - 9.9788\times10^{-8} t^2 + 9.0101\times10^{-11} t^3\right) \\
&= -1.0023\times10^{-5} - 1.9958\times10^{-7} t + 2.7030\times10^{-10} t^2, \quad 1 \le t \le 1000
\end{aligned}$$

Example 2-Direct Fit Polynomials cont.

$$R'(50) = -1.0023\times10^{-5} - 1.9958\times10^{-7}(50) + 2.7030\times10^{-10}(50)^2$$
$$= -1.9326\times10^{-5}$$

Using the same function, we can also calculate the value of $R(t)$ at $t = 50$.

$$R(t) = 0.99991 - 1.0023\times10^{-5}t - 9.9788\times10^{-8}t^2 + 9.0101\times10^{-11}t^3,\ \ 1 \le t \le 1000$$

$$R(50) = 0.99991 - 1.0023\times10^{-5}(50) - 9.9788\times10^{-8}(50)^2 + 9.0101\times10^{-11}(50)^3$$
$$= 0.99917$$

The failure rate is then $h(t) = -\dfrac{R'(t)}{R(t)}$

$$= \frac{\left(-1.9326\times10^{-5}\right)}{0.99917}$$
$$= 1.9343\times10^{-5}$$

Lagrange Polynomial

In this method, given $(x_1, y_1), \ldots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating equation (2) gives

Lagrange Polynomial Cont.

$$f_2'(x)=\frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x)=\frac{2}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{2}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{2}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Example 3

The failure rate $h(t)$ of a direct methanol fuel cell (DMFC) is given by the formula

$$h(t) = \frac{R'(t)}{R(t)}$$

Where $R(t)$ is the reliability at a certain time t, and the values of the reliability are given in Table 3.

Table 3 Reliability of DMFC system.

t (hrs)	0	1	10	100	1000	2000	3000	4000	5000
$R(t)$	1	0.9999	0.9998	0.9980	0.9802	0.9609	0.9419	0.9233	0.9050

Determine the value of the failure rate at $t = 50$ hours using the second order Lagrangian polynomial interpolation for reliability.

Example 3 Cont.

Solution

For second order Lagrangian polynomial interpolation, we choose the reliability given by

$$R(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)R(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)R(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)R(t_2)$$

Since we want to find the reliability at $t=50$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t=50$ that also bracket $t=50$ to evaluate it.
The three points are $t_0=1$, $t_1=10$, and $t_2=100$.

Differentiation the above equation gives.

$$R'(t)=\frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}R(t_0)+\frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}R(t_1)+\frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}R(t_2)$$

Example 3 Cont.

Hence

$$R'(50)=\frac{2(50)-(10+100)}{(1-10)(1-100)}(0.9999)+\frac{2(50)-(1+100)}{(10-1)(10-100)}(0.9998)+\frac{2(50)-(1+10)}{(100-1)(100-10)}(0.9980)$$

$$=-1.9102\times 10^{-5}$$

We must also find the value of $R(t)$ at $t=50$.

$$R(50)=\left(\frac{50-10}{1-10}\right)\left(\frac{50-100}{1-100}\right)(0.9999)+\left(\frac{50-1}{10-1}\right)\left(\frac{50-100}{10-100}\right)(0.9998)$$

$$+\left(\frac{50-1}{100-1}\right)\left(\frac{50-10}{100-10}\right)(0.9980)$$

$$=0.99918$$

Example 3 Cont.

The failure rate is then

$$h(t) = -\frac{R'(t)}{R(t)}$$

$$= -\frac{\left(-1.9102\times10^{-5}\right)}{0.99918}$$

$$= 1.9118\times10^{-5}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/discrete_02dif.html

THE END

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