Euler Method

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Transforming Numerical Methods Education for STEM Undergraduates

Euler Method

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Euler's Method

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

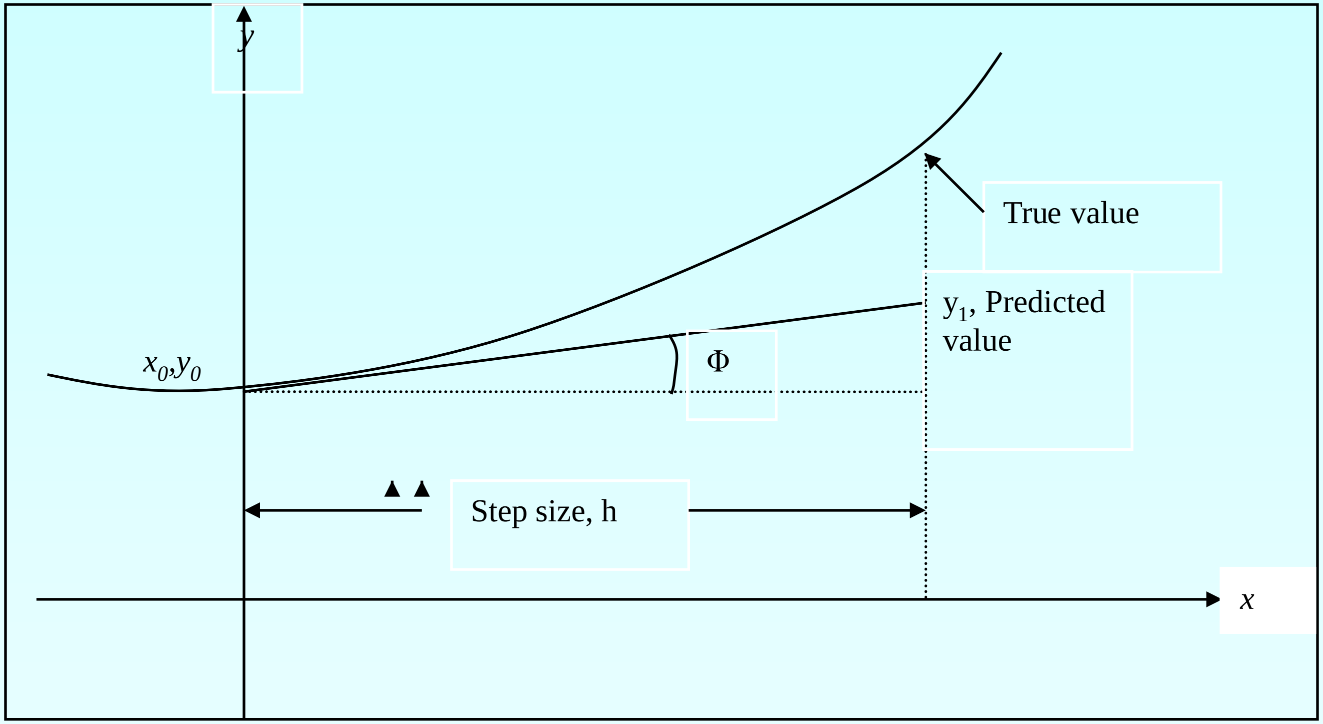

Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$h = x_{i+1} - x_i$$

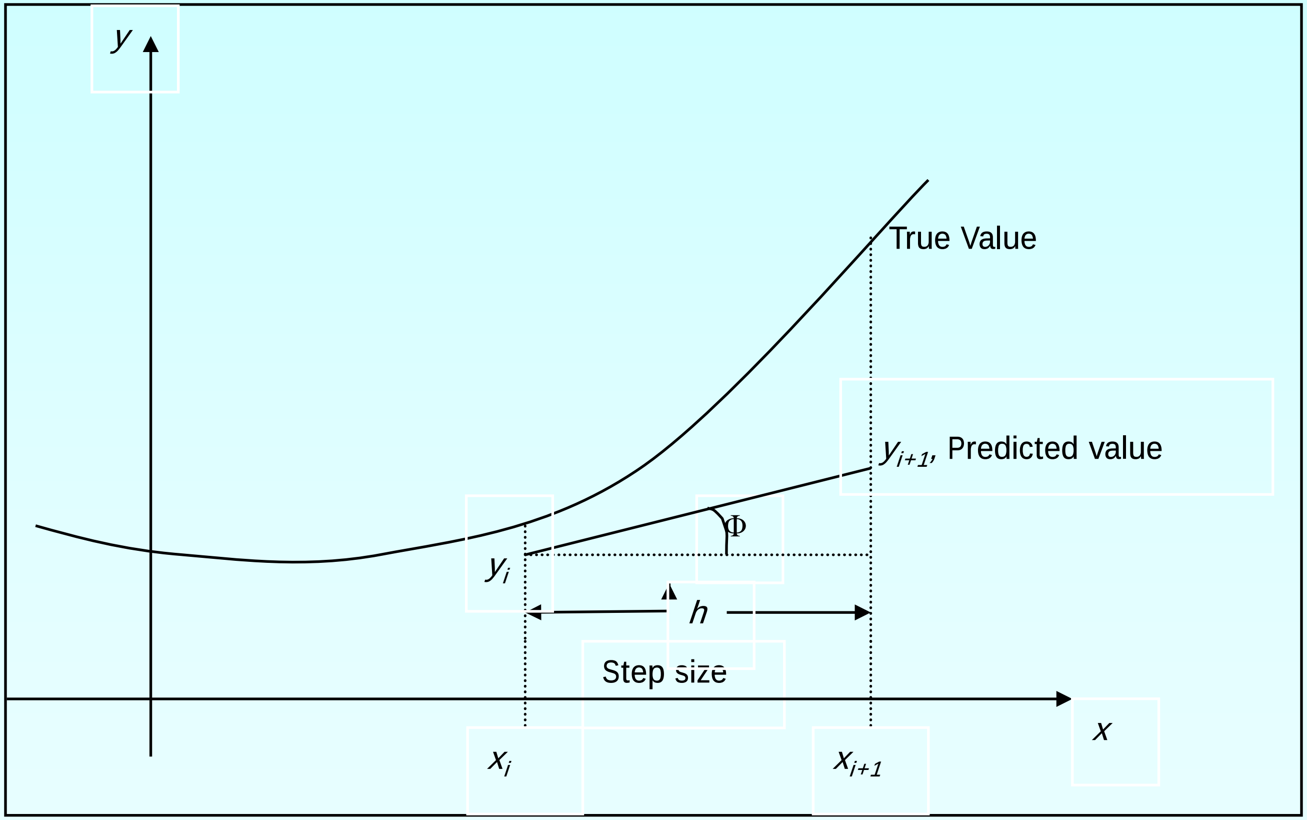

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The open loop response, that is, the speed of the motor to a voltage input of 20 V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w$$

If the initial speed is zero $(w(0) = 0)$, and using Euler's method, what is the speed at $t = 0.8\,\mathrm{s}$? Assume a step size of $h = 0.4\,\mathrm{s}$.

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(t, w) = 1000 - 3w$$

$$w_{i+1} = w_i + f(t_i, w_i)h$$

Solution

Step 1: For $i = 0, \quad t_0 = 0, \quad w_0 = 0$

$$
\begin{aligned}
w_1 &= w_0 + f(t_0, w_0)h \\
&= 0 + f(0,0) \times 0.4 \\
&= 0 + (1000 - 3(0))0.4 \\
&= 0 + 1000 \times 0.4 \\
&= 400 \, \text{rad/s}
\end{aligned}
$$

w_1 is the approximate speed of the motor at

$$t = t_1 = t_0 + h = 0 + 0.4 = 0.4 \, s$$

$$w(0.4) \approx w_1 = 400 \, \text{rad/s}$$

Solution Cont

Step 2: For $i = 1, \quad t_1 = 0.4, \quad w_2 = 400$

$$\begin{aligned} w_2 &= w_1 + f(t_1, w_1)h \\ &= 400 + f(0.4,\ 400)0.4 \\ &= 400 + (1000 - 3(400))0.4 \\ &= 400 + (-200)0.4 \\ &= 320\,\text{rad/s} \end{aligned}$$

w_2 is the approximate speed of the motor at

$$t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8\,\text{s}$$

$$x(0.8) \approx w_2 = 320\ \text{rad/s}$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right)e^{-3t}$$

The solution to this nonlinear equation at $t = 0.8$ s is

$$w(0.8) = 303.09 \text{ rad/s}$$

Comparison of Exact and Numerical Solutions

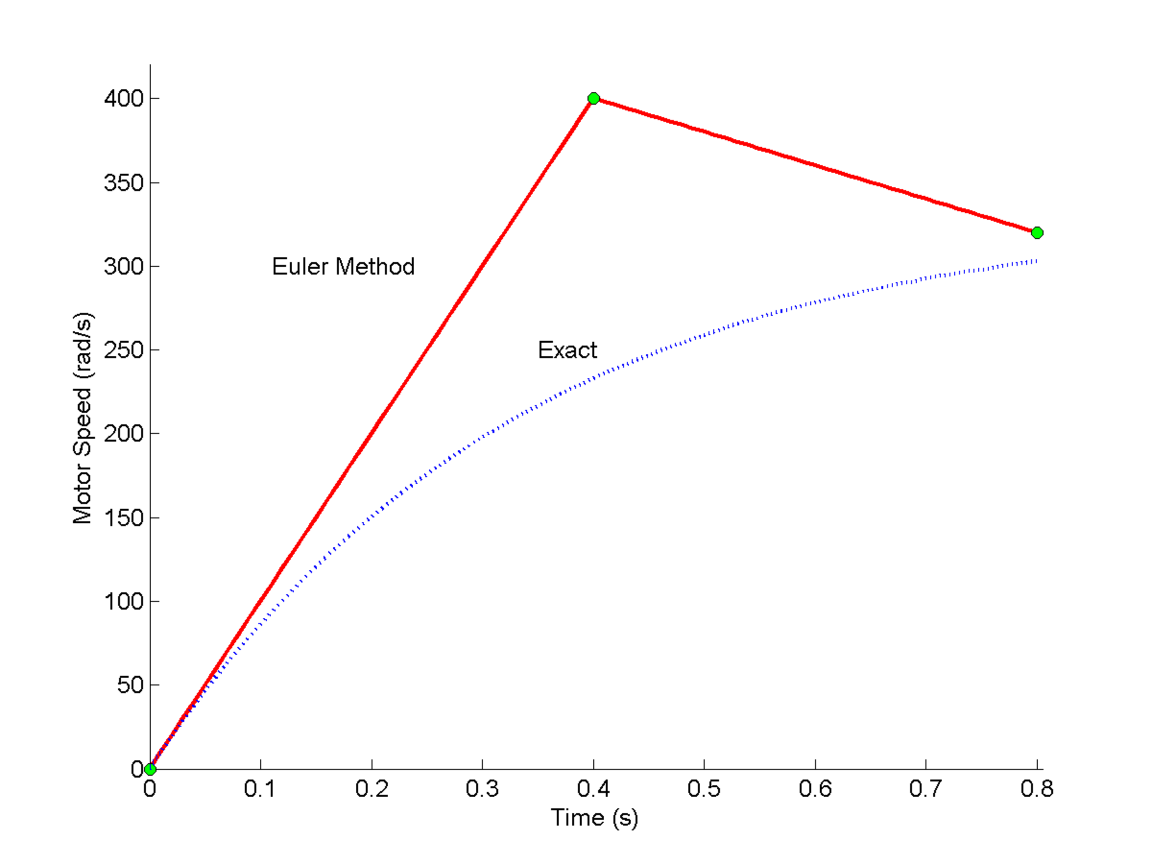

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1 Speed of motor at 0.8 seconds as a function of step size, *h*

Step size, h	$w(0.8)$	E_t	$\lvert\in_t\rvert\ \%$
0.8	800	−496.91	163.95
0.4	320	−16.906	5.5778
0.2	324.8	−21.706	7.1615
0.1	314.18	−11.023	3.6370
0.05	308.58	−5.4890	1.8110

Comparison with exact results

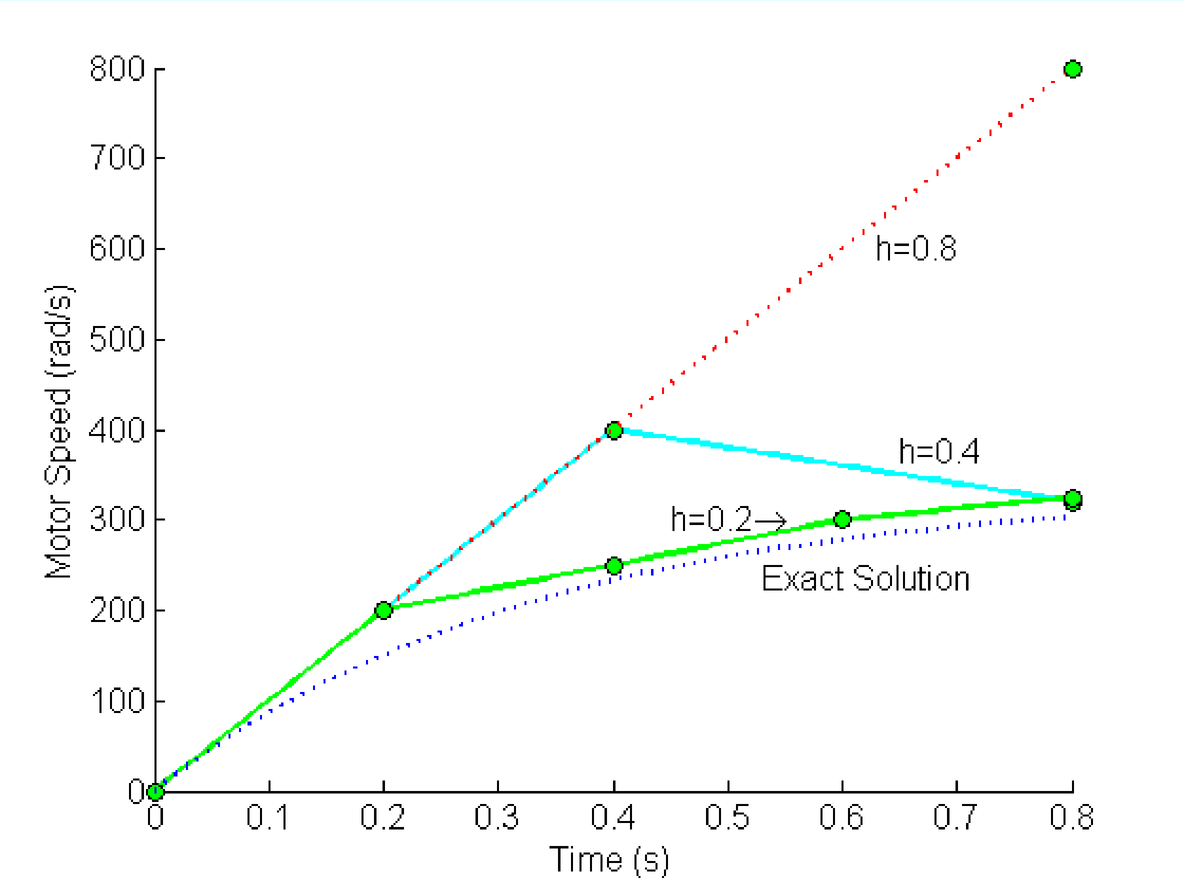

Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

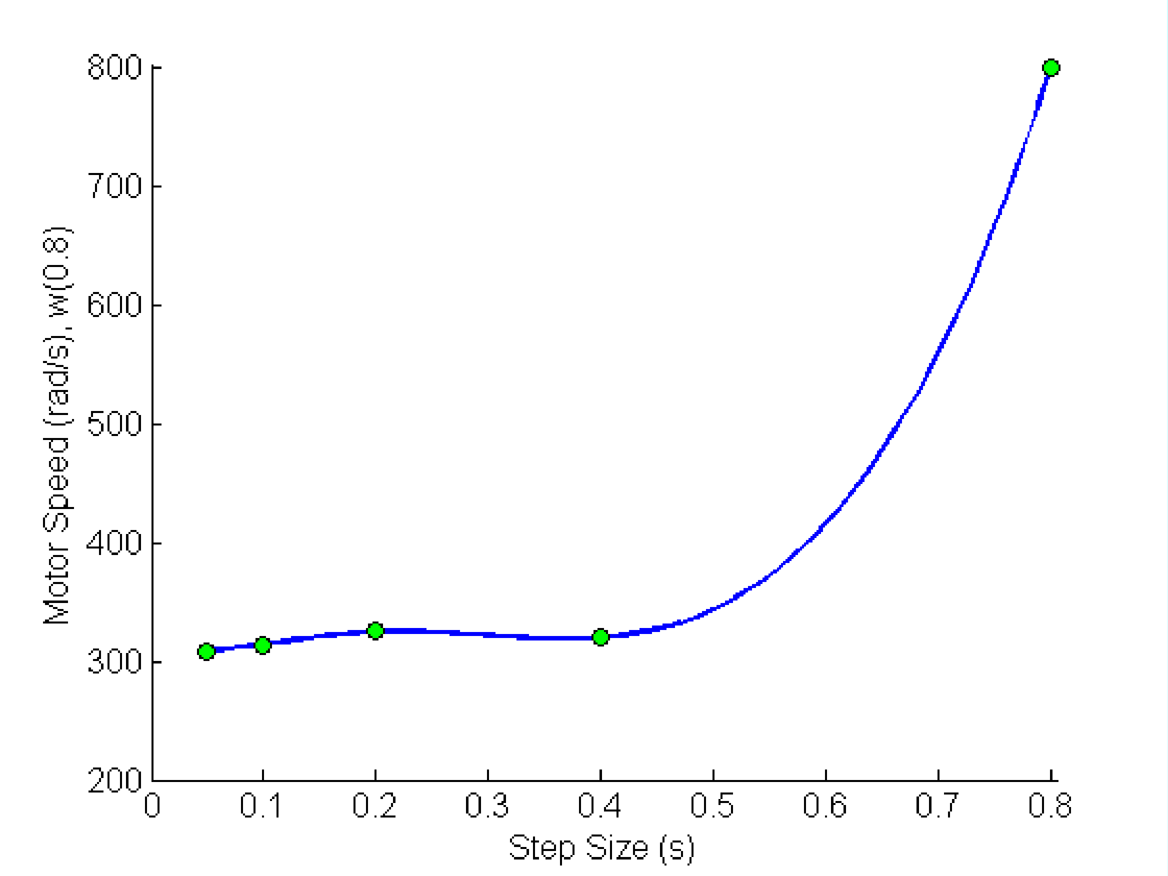

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \qquad E_t \propto h^2$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/euler_method.html

THE END

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