Runge 2nd Order Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 2nd Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

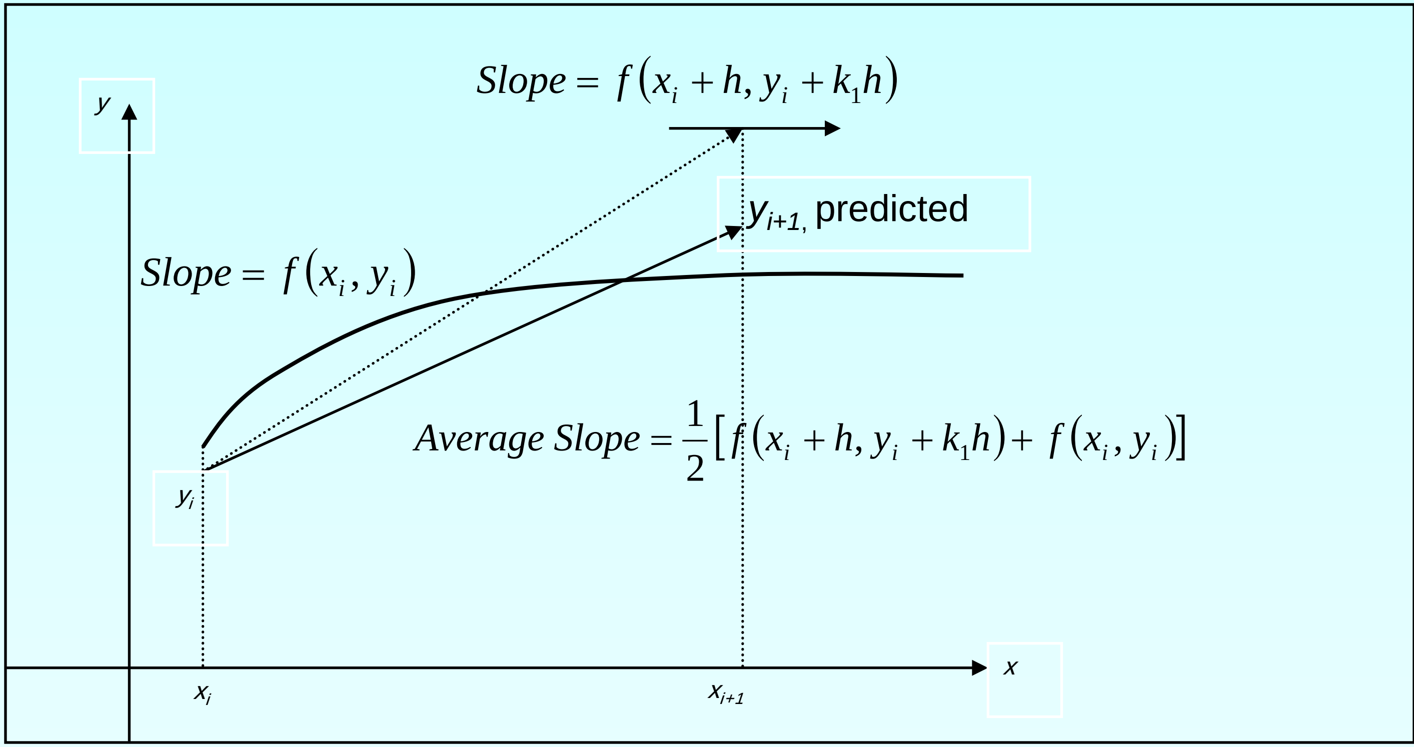

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The open loop response, that is, the speed of the motor to a voltage input of 20 V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w$$

If the initial speed is zero; use the Runge-Kutta 2nd order method and a step size of $h = 0.4\,\text{s}$ to find the speed at $t = 0.8\ \text{s}$.

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(t, w) = 1000 - 3w$$

$$w_{i+1} = w_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0,\ t_0 = 0,\ w_0 = 0$

$$k_1 = f\left(t_0, w_o\right) = f\left(0,\ 0\right) = 1000 - 3\left(0\right) = 1000$$

$$k_2 = f\left(t_0 + h,\ w_0 + k_1 h\right) = f\left(0 + 0.4,\ 0 + \left(1000\right)0.4\right) = f\left(0.4,\ 400\right) = 1000 - 3\left(400\right) = -200$$

$$\begin{aligned} w_1 &= w_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 0 + \left(\frac{1}{2}\left(1000\right) + \frac{1}{2}\left(-200\right)0.4\right) \\ &= 0 + \left(500 - 100\right)0.4 \\ &= 160 \text{ rad/s} \end{aligned}$$

Solution Cont

Step 2: $i = 1,\ t_1 = t_0 + h = 0 + 0.4 = 0.4,\ \ w_1 = 160$

$$k_1 = f(t_1, w_1) = f(0.4, 160) = 1000 - 3(160) = 520$$

$$k_2 = f(t_1 + h,\ w_1 + k_1 h) = f(0.4 + 0.4,\ 160 + (520)0.4)$$
$$= f(0.8,\ 368) = 1000 - 3(368) = -104$$

$$w_2 = w_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 160 + \left(\frac{1}{2}(520) + \frac{1}{2}(-104)\right)0.4$$
$$= 160 + (208)0.4$$
$$= 243.2 \text{ rad/s}$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right)e^{-3t}$$

The solution to this nonlinear equation at t=3 minutes is

$$w(0.8) = 303.09 \text{ rad/s}$$

Comparison with exact results

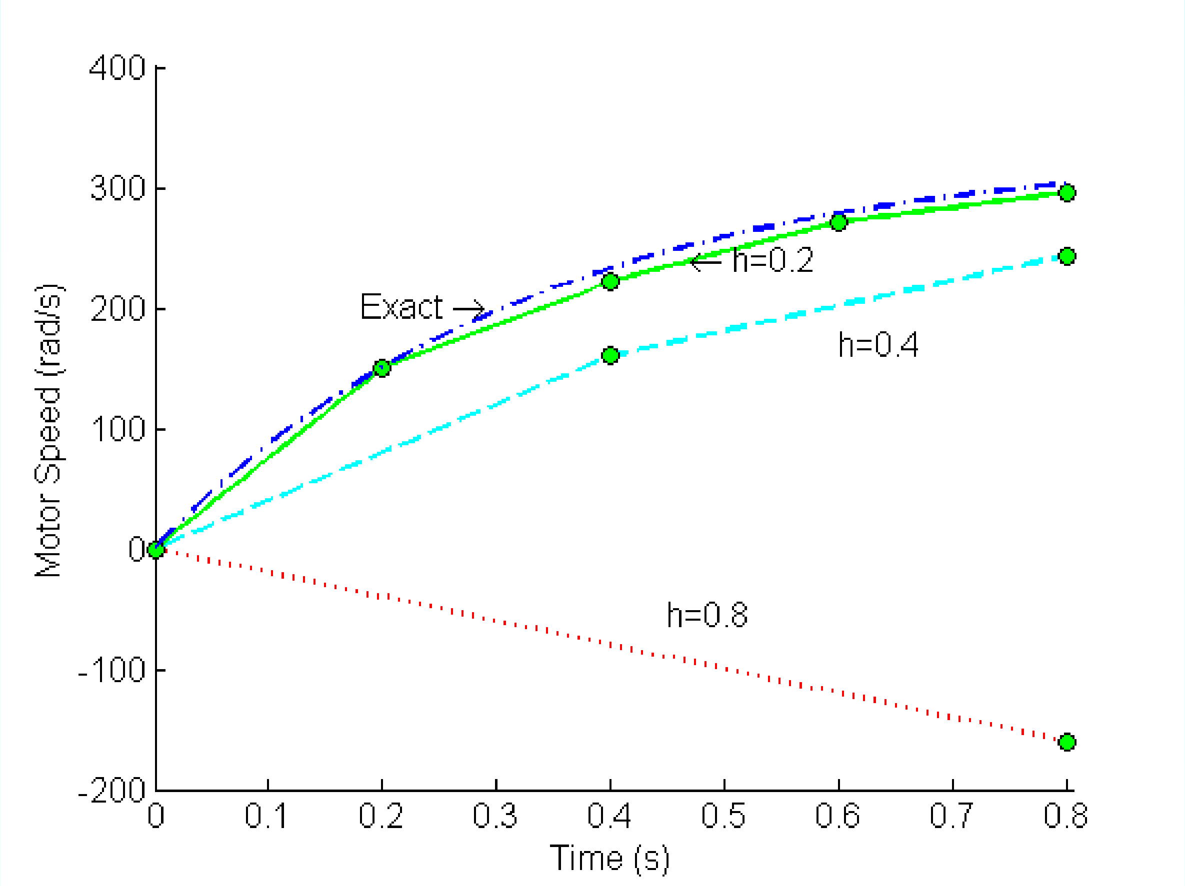

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1 Effect of step size for Heun's method

Step size, h	$x(3)$	E_t	$\lvert\in_t\rvert\%$
0.8	−160.00	463.09	152.79
0.4	243.20	59.894	19.761
0.2	295.61	7.4823	2.4687
0.1	301.70	1.3929	0.45954
0.05	302.79	0.30613	0.10100

$w(0.8) = 303.09 \text{ rad/s}$ (exact)

Effects of step size on Heun's Method

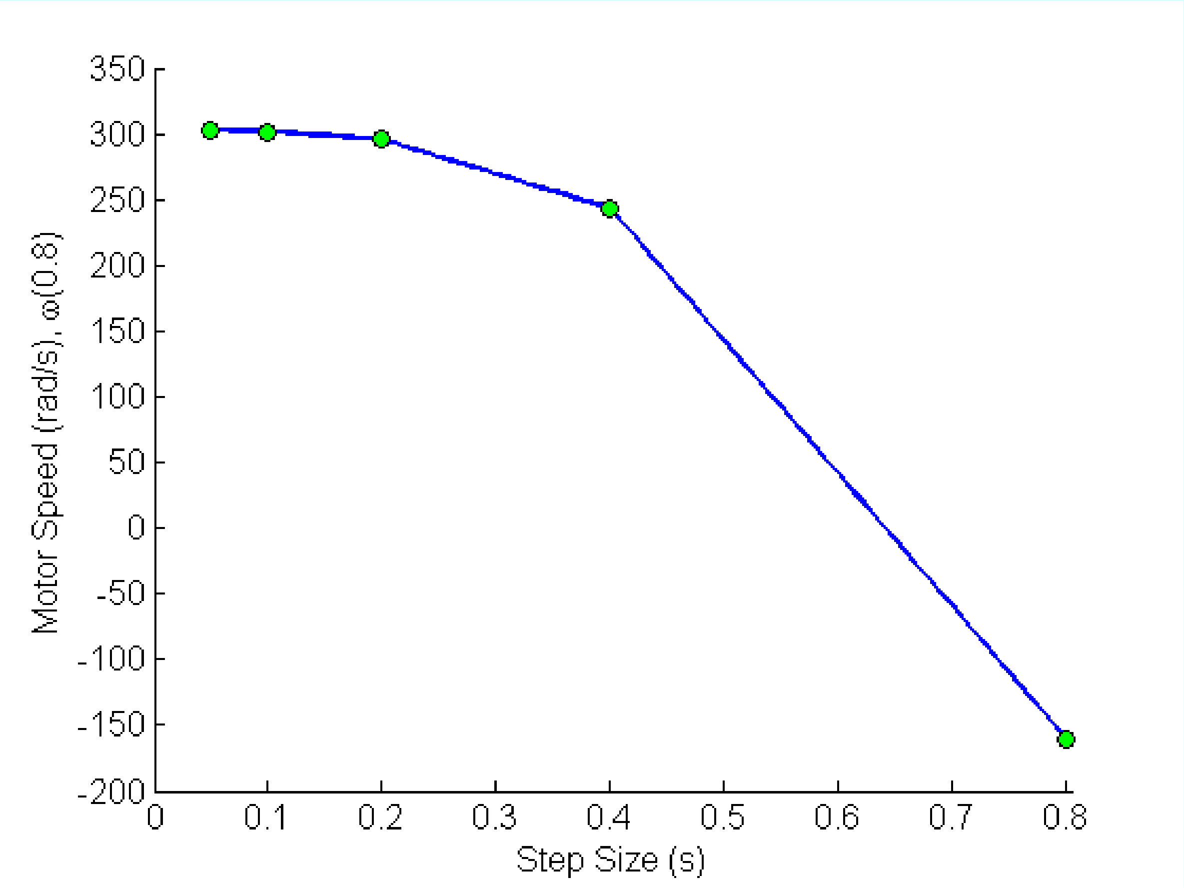

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$w(0.8)$			
	Euler	Heun	Midpoint	Ralston
0.8	800	−160.00	−160.00	−160.00
0.4	320	243.20	243.20	243.20
0.2	324.8	295.61	295.61	295.61
0.1	314.11	301.70	301.70	301.70
0.05	308.58	302.79	302.79	302.79

$w(0.8) = 303.09\ \text{rad/s}$ (exact)

Comparison of Euler and Runge-Kutta 2^{nd} Order Methods

Table 2 Comparison of Euler and the Runge-Kutta methods

Step size, h	$\lvert \in_t \rvert \%$			
	Euler	Heun	Midpoint	Ralston
0.8	163.94	152.79	152.79	152.79
0.4	5.5792	19.760	19.760	19.760
0.2	7.1629	2.4679	2.4679	2.4679
0.1	3.6359	0.45861	0.45861	0.45861
0.05	1.8113	0.098981	0.098981	0.098981

$w(0.8) = 303.09 \text{ rad/s}$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

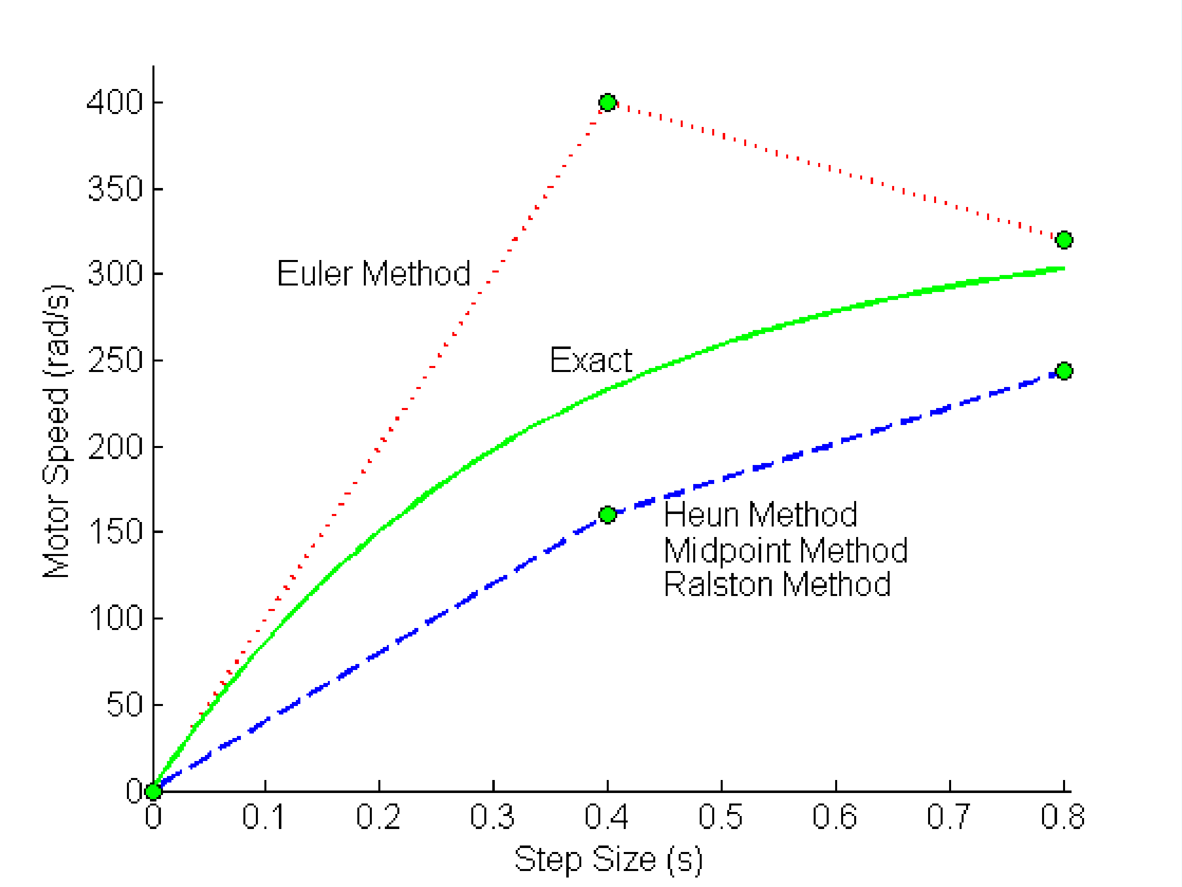

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

THE END

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