Runge 4^{th} Order Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 4th Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The open loop response, that is, the speed of the motor to a voltage input of 20 V, assuming a system without damping is

$$20 = (0.02)\frac{dw}{dt} + (0.06)w$$

If the initial speed is zero $\left(w(0) = 0\right)$, and using the Runge-Kutta 4th order method, what is the speed at t = 0.8 s? Assume a step size of h = 0.4 s.

$$\frac{dw}{dt} = 1000 - 3w$$

$$f(t, w) = 1000 - 3w$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: For $i = 0, \quad t_0 = 0 \quad w_0 = 0$

$$k_1 = f\left(t_0, w_0\right) = f\left(0,\ 0\right) = 1000 - 3\left(0\right) = 1000$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(0.4),\ 0 + \frac{1}{2}(1000)0.4\right) = f\left(0.2,\ 200\right) = 1000 - 3\left(200\right) = 400$$

$$k_3 = f\left(t_0 + \frac{1}{2}h,\ w_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(0.4),\ 0 + \frac{1}{2}(400)0.4\right) = f\left(0.2,\ 80\right) = 1000 - 3\left(80\right) = 760$$

$$k_4 = f\left(t_0 + h,\ w_0 + k_3 h\right) = f\left(0 + (0.4),\ 0 + (760)0.4\right) = f\left(0.4,\ 304\right) = 1000 - 3\left(304\right) = 88$$

Solution Cont

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 0 + \frac{1}{6}\left(1000 + 2(400) + 2(760) + 88\right)0.4$$

$$= 0 + \frac{1}{6}(3408)0.4$$

$$= 227.2 \text{ rad/s}$$

w_1 is the approximate speed of the motor at

$$t = t_1 = t_0 + h = 0 + 0.4 = 0.4\,\text{s}$$

$$w(0.4) \approx w_1 = 227.2 \text{ rad/s}$$

Solution Cont

Step 2: $i = 1,\ t_1 = 0.8,\ w_1 = 227.2$

$$k_1 = f(t_1, w_1) = f(0.4,\ 227.2) = 1000 - 3(227.2) = 318.4$$

$$k_2 = f\left(t_1 + \frac{1}{2}h,\ w_1 + \frac{1}{2}k_1 h\right) = f\left(0.4 + \frac{1}{2}(0.4),\ 227.2 + \frac{1}{2}(318.4)0.4\right)$$

$$= f(0.6,\ 290.88) = 1000 - 3(290.88) = 127.36$$

$$k_3 = f\left(t_1 + \frac{1}{2}h,\ w_1 + \frac{1}{2}k_2 h\right) = f\left(0.4 + \frac{1}{2}(0.4),\ 227.2 + \frac{1}{2}(127.36)0.4\right)$$

$$= f(0.6,\ 252.67) = 1000 - 3(252.67) = 241.98$$

$$k_4 = f(t_1 + h,\ w_1 + k_3 h) = f(0.4 + 0.4,\ 227.2 + (241.98)0.4)$$

$$= f(0.8,\ 323.99) = 1000 - 3(323.99) = 28.019$$

Solution Cont

$$w_2 = w_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 227.2 + \frac{1}{6}\left(318.4 + 2\left(127.36\right) + 2\left(241.98\right) + 28.019\right)0.4$$

$$= 227.2 + \frac{1}{6}\left(1085.1\right)0.4$$

$$= 299.54 \text{ rad/s}$$

w_2 is the approximate speed of the motor at

$$t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8\text{s}$$

$$w\left(0.8\right) \approx w_2 = 299.54 \text{ rad/s}$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$w(t) = \left(\frac{1000}{3}\right) - \left(\frac{1000}{3}\right)e^{-3t}$$

The solution to this nonlinear equation at t=0.8 seconds is

$$w(0.8) = 303.09 \ \text{rad/s}$$

Comparison with exact results

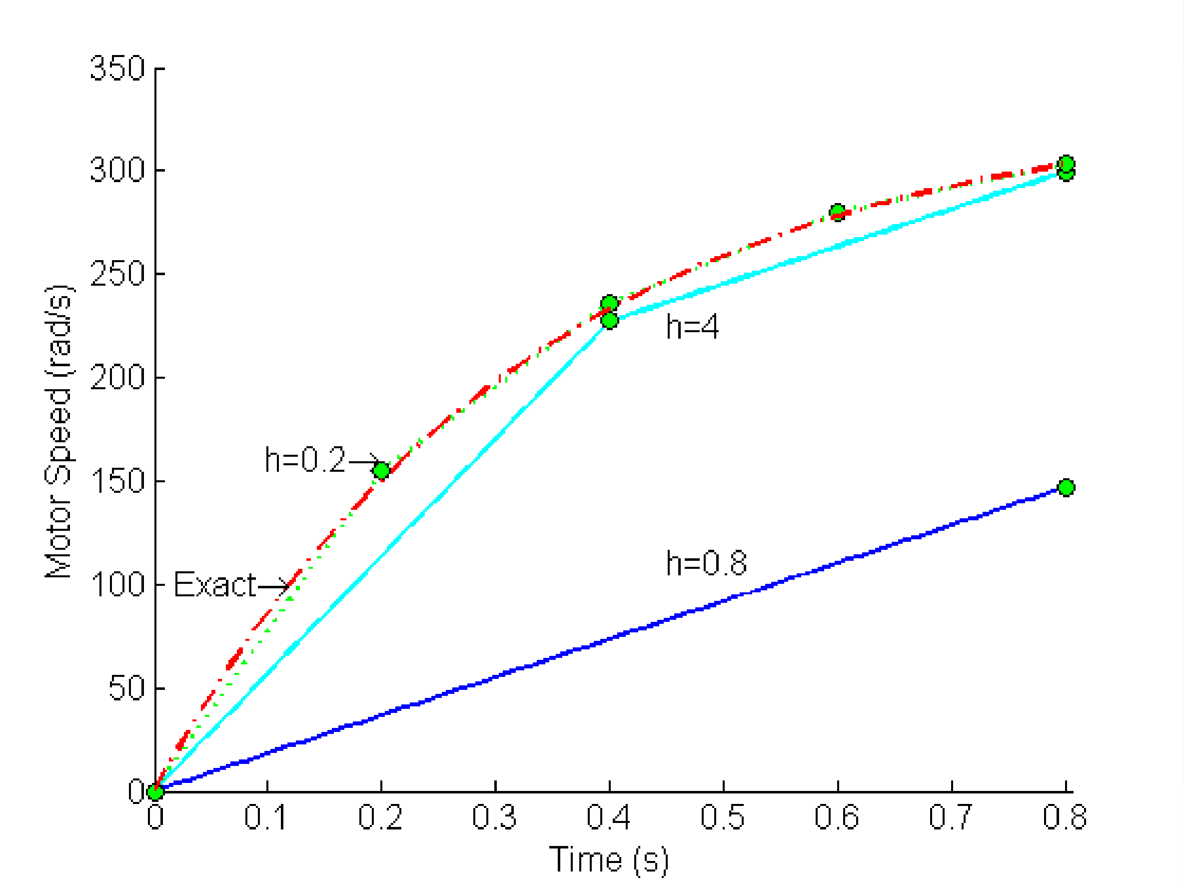

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1 Values of speed of the motor at 0.8 seconds for different step sizes

Step size, h	$w(0.8)$	E_t	$\|\in_t\|$ %
0.8	147.20	155.89	51.434
0.4	299.54	3.5535	1.1724
0.2	302.96	0.12988	0.042852
0.1	303.09	0.0062962	0.0020773
0.05	303.09	0.00034702	0.00011449

$w(0.8) = 303.09$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

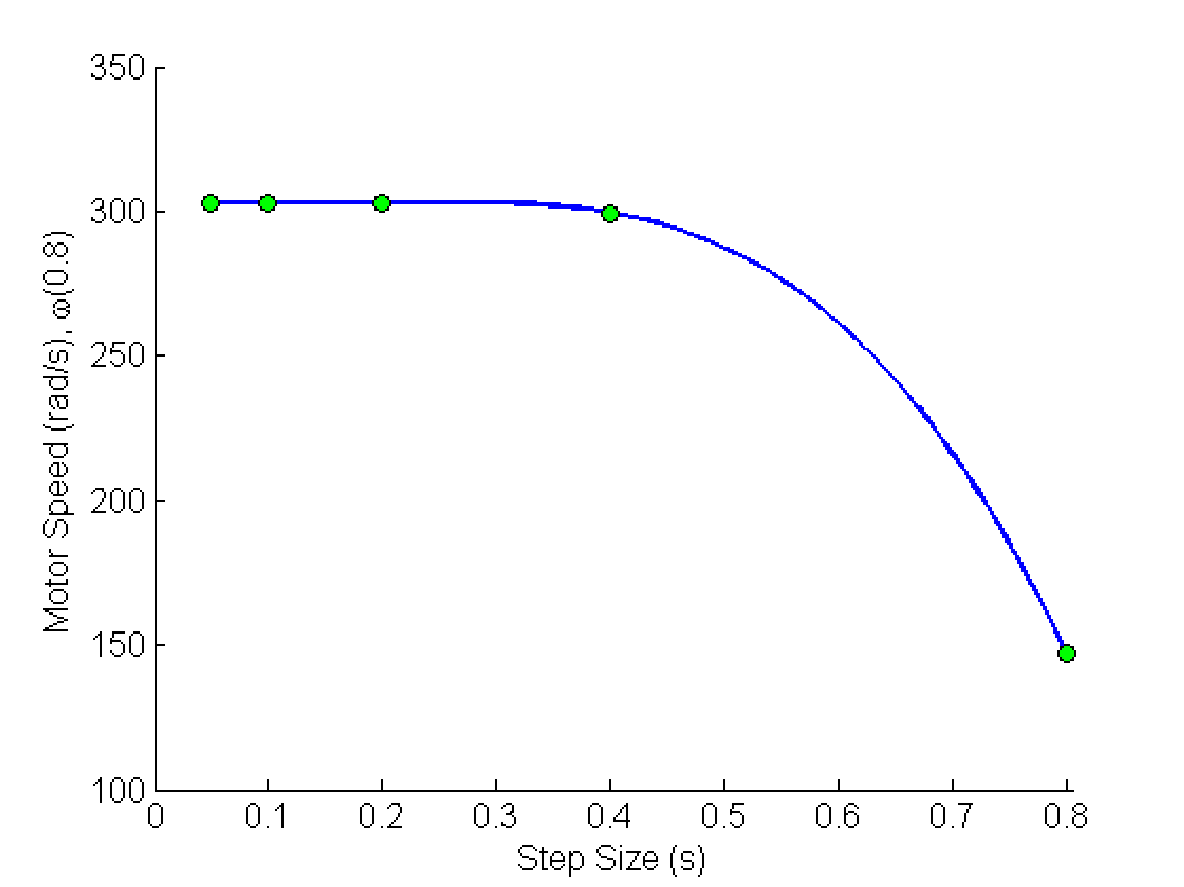

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

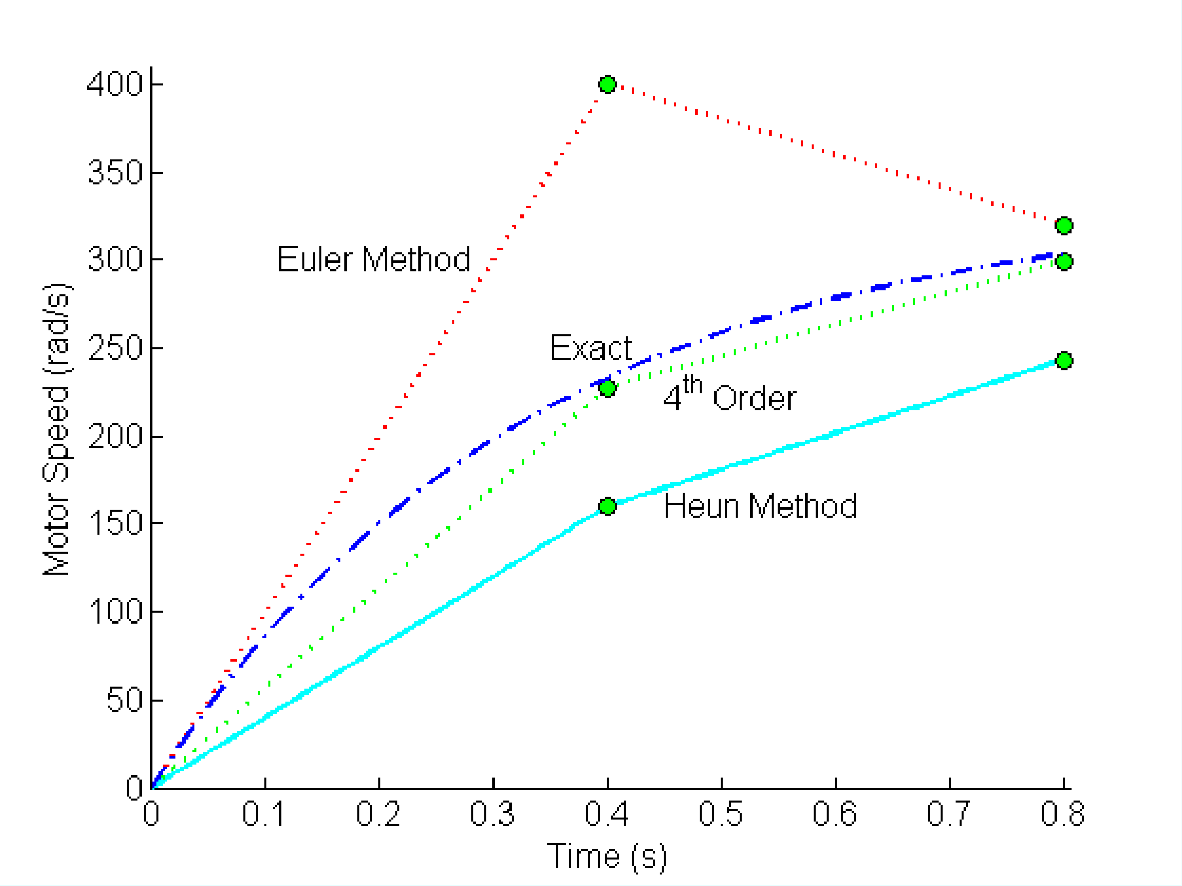

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

THE END

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