Differentiation-Discrete Functions

Mechanical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Differentiation – ContinuousDiscrete Functions

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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

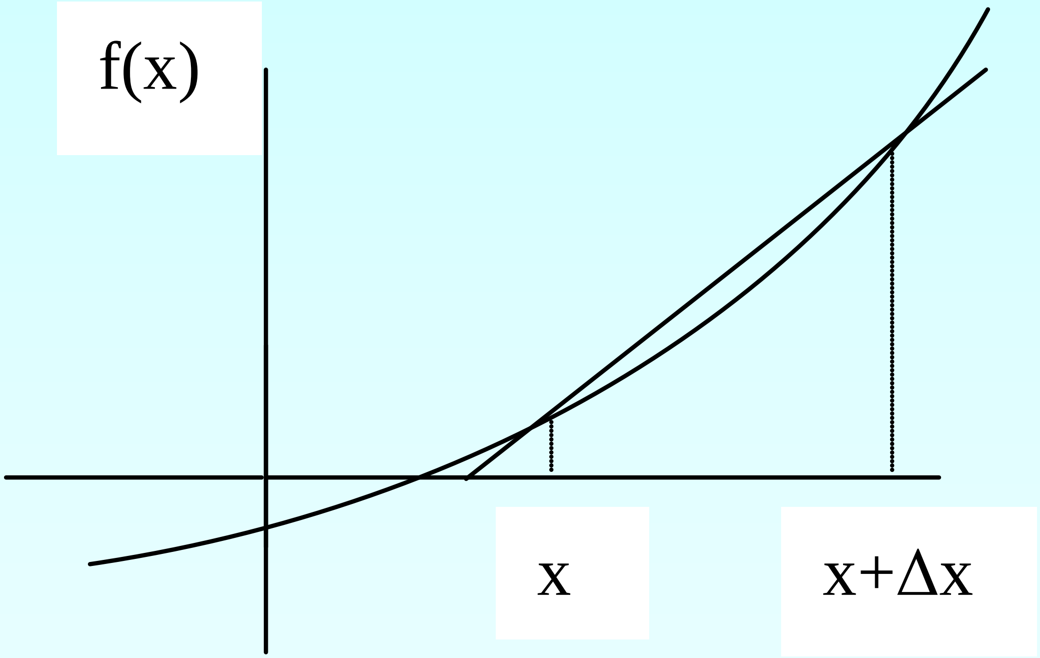

Figure 1 Graphical Representation of forward difference approximation of first derivative.

Example 1

To find contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 1.

a) Is the rate of change of coefficient of thermal expansion with respect to temperature more at $T = 80\ ^\circ\mathrm{F}$ than at $T = -340\ ^\circ\mathrm{F}$?

b) The data given in Table 1 can be regressed to $\alpha = a_0 + a_1\mathrm{T} + a_2\mathrm{T}^2$ to get $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}\mathrm{T} - 1.2215\times10^{-11}\mathrm{T}^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80\ ^\circ\mathrm{F}$ and at $T = -340\ ^\circ\mathrm{F}$.

Example 1 Cont.

Table 1 Coefficient of thermal expansion as a function of temperature.

Temperature, T (°F)	Coefficient of thermal expansion, α (in/in/°F)
80	6.47×10^{-6}
40	6.24×10^{-6}
−40	5.72×10^{-6}
−120	5.09×10^{-6}
−200	4.30×10^{-6}
−280	3.33×10^{-6}
−340	2.45×10^{-6}

Example 1 Cont.

Solution

a) Using the forward divided difference approximation method at $T = 80^0 F$,

$$\frac{d\alpha(T_i)}{dT} \approx \frac{\alpha(T_{i+1}) - \alpha(T_i)}{\Delta T}$$

$$T_i = 80$$

$$\Delta T = -40$$

$$\begin{aligned} T_{i+1} &= T_i + \Delta T \\ &= 80 + (-40) \\ &= 40 \end{aligned}$$

Example 1 Cont.

$$\frac{d\alpha(80)}{dT} \approx \frac{\alpha(40)-\alpha(80)}{-40}$$
$$\approx \frac{6.24\times10^{-6}-6.47\times10^{-6}}{40}$$
$$\approx 5.75\times10^{-9}\ \text{in/in/°F}^2$$

Using backwards divided difference approximation method at $T=-340^0F$,

$$\frac{d\alpha(T_i)}{dT} \approx \frac{\alpha(T_i)-\alpha(T_{i-1})}{\Delta T}$$

$$T_i = -340$$
$$\Delta T = -60$$

$$T_{i-1} = T_i - \Delta T$$
$$= -340-(-60)$$
$$= -280$$

Example 1 Cont.

$$\frac{d\alpha(-340)}{dT} \approx \frac{\alpha(-340)-\alpha(-280)}{-60}$$
$$\approx \frac{2.45\times10^{-6}-3.33\times10^{-6}}{-60}$$
$$\approx 0.14667\times10^{-7}\ \text{in/in/°F}^2$$

From the above two results it is clear that the rate of change of coefficient of thermal expansion is more at $T = 80^0F$ than at $T = -340^0F$.

b) Given $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}T - 1.2215\times10^{-11}T^2$

$$\frac{d\alpha}{dT} = 6.279\times10^{-9} - 2.443\times10^{-11}T$$

$$\frac{d\alpha(80)}{dT} = 6.279\times10^{-9} - 2.443\times10^{-11}(80)$$
$$= 4.3246\times10^{-9}\ \text{in/in/°F}^2$$

Example 1 Cont.

$$\frac{d\alpha(-340)}{dT} = 6.279\times10^{-9} - 2.443\times10^{-11}(-340)$$
$$= 0.14585\times10^{-7} \text{ in/in/°F}^2$$

Table 2 Summary of change in coefficient of thermal expansion using different approximations.

Temperature, T_i	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	Divided Difference Approximation	2^{nd} Order Polynomial Regression
80^0F	$5.75\times10^{-9}\, in/in/^oF^2$	$4.3246\times10^{-9}\, in/in/^oF^2$
-340^0F	$0.14667\times10^{-7}\, in/in/^oF^2$	$0.14585\times10^{-7}\, in/in/^oF^2$

Direct Fit Polynomials

In this method, given $'n+1'$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1}x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1}x^{n-2} + na_n x^{n-1}$$

Similarly other derivatives can be found.

Example 2-Direct Fit Polynomials

To find contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 3.

a) Using the third order polynomial interpolant, find the change in coefficient of thermal expansion at $T = 80°\mathrm{F}$ and $T = -340°\mathrm{F}$.

b) The data given in Table 3 can be regressed to $\alpha = a_0 + a_1 T + a_2 T^2$ to get $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}\mathrm{T} - 1.2215\times10^{-11}\mathrm{T}^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80°\mathrm{F}$ and $T = -340°\mathrm{F}$.

Example 2 Cont.

Table 3 Coefficient of thermal expansion as a function of temperature.

Temperature, T (°F)	Coefficient of thermal expansion, α (in/in/°F)
80	6.47×10^{-6}
40	6.24×10^{-6}
-40	5.72×10^{-6}
-120	5.09×10^{-6}
-200	4.30×10^{-6}
-280	3.33×10^{-6}
-340	2.45×10^{-6}

Example 2-Direct Fit Polynomials cont.

Solution

For the third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(t) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

a) Change in Thermal Expansion Coefficient at $80^0 F$:
Since we want to find the rate of change in the thermal expansion coefficient at $T = 80^0 F$, and we are using a third order polynomial, we need to choose the four points closest to $T = 80^0 F$ that also bracket $T = 80^0 F$ to evaluate it.

The four points are $T_0 = 80\ °\mathrm{F}$, $T_1 = 40\ °\mathrm{F}$, $T_2 = -40\ °\mathrm{F}$, and $T_3 = -120\ °\mathrm{F}$.

$$T_o = 80,\ \alpha(T_o) = 6.47 \times 10^{-6}$$

$$T_1 = 40,\ \alpha(T_1) = 6.24 \times 10^{-6}$$

$$T_2 = -40,\ \alpha(T_2) = 5.72 \times 10^{-6}$$

$$T_3 = -120,\ \alpha(T_3) = 5.09 \times 10^{-6}$$

Example 2-Direct Fit Polynomials cont.

such that

$$\alpha(80) = 6.47\times10^{-6} = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3$$

$$\alpha(40) = 6.24\times10^{-6} = a_0 + a_1(40) + a_2(40)^2 + a_3(40)^3$$

$$\alpha(-40) = 5.72\times10^{-6} = a_0 + a_1(-40) + a_2(-40)^2 + a_3(-40)^3$$

$$\alpha(-120) = 5.09\times10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 & 512000 \\ 1 & 40 & 1600 & 64000 \\ 1 & -40 & 1600 & -64000 \\ 1 & -120 & 14400 & -1728000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47\times10^{-6} \\ 6.24\times10^{-6} \\ 5.72\times10^{-6} \\ 5.09\times10^{-6} \end{bmatrix}$$

Example 2-Direct Fit Polynomials cont.

Solving the above four equations gives

$$a_0 = 0.59915 \times 10^{-5}$$

$$a_1 = 0.64813 \times 10^{-8}$$

$$a_2 = -0.71875 \times 10^{-11}$$

$$a_3 = 0.11719 \times 10^{-13}$$

Hence

$$\alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

$$= 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2 + 0.11719 \times 10^{-13} T^3,$$

$$-120 \le T \le 80$$

Example 2-Direct Fit Polynomials cont.

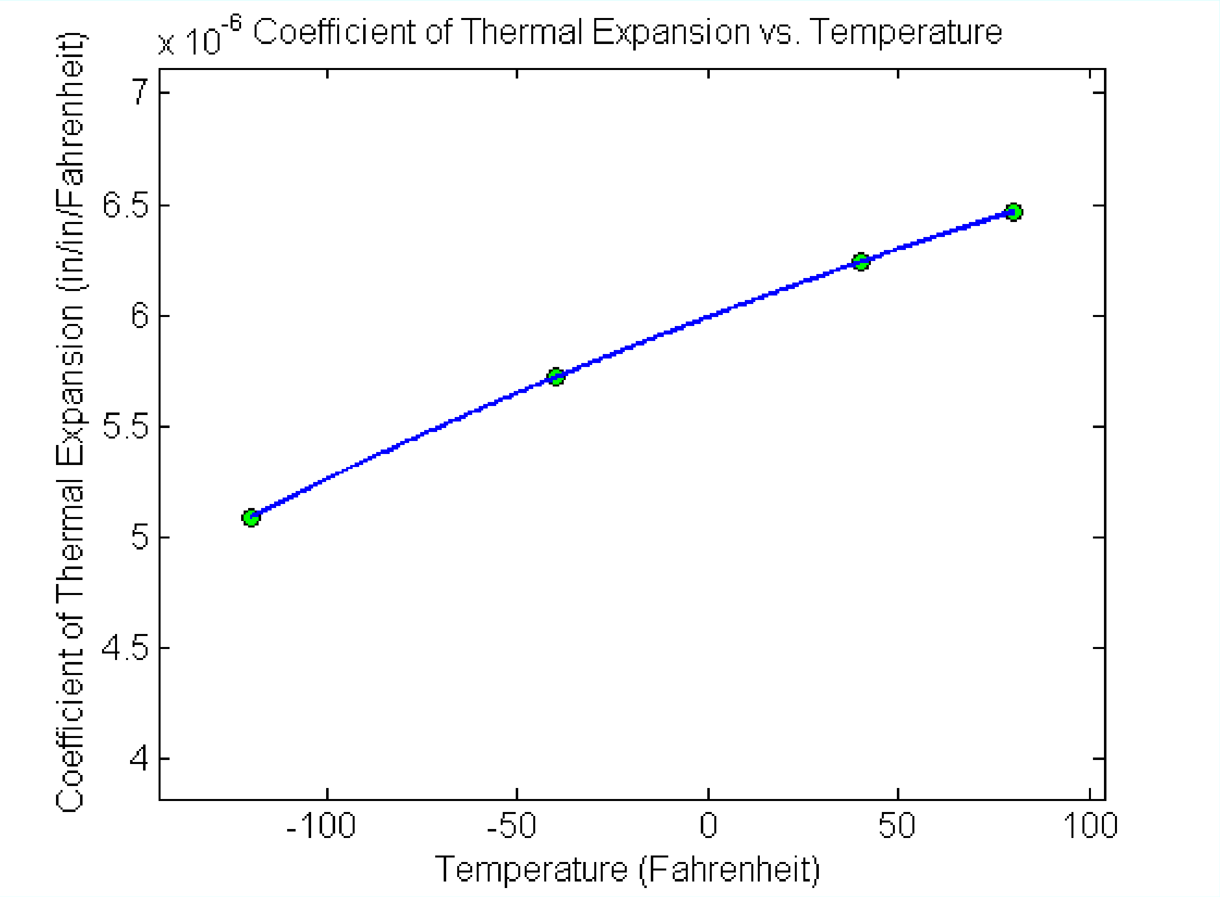

Figure 2 Graph of coefficient of thermal expansion vs. temperature.

Example 2-Direct Fit Polynomials cont.

The change in coefficient of thermal expansion at $T = 80^0 F$ is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dt}\alpha(T)\Big|_{T=80}$$

Given that

$$\alpha(T) = 0.59915\times10^{-5} + 0.64813\times10^{-8}T - 0.71875\times10^{-11}T^2 + 0.11719\times10^{-13}T^3, \quad -120 \le T \le 80$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT}\alpha(T)$$

$$= \frac{d}{dt}\left(0.59915\times10^{-5} + 0.64812\times10^{-8}T - 0.71875\times10^{-11}T^2 + 0.11719\times10^{-13}T^3\right)$$

$$= 0.64813\times10^{-8} - 1.4375\times10^{-11}T + 0.35157\times10^{-13}T^2, \ -120 \le T \le 80$$

$$\frac{d\alpha(80)}{dT} = 0.64813\times10^{-8} - 1.4375\times10^{-11}(80) + 0.35157\times10^{-13}T(80)^2$$

$$= 5.5563\times10^{-9}\,\text{in/in/°F}^2$$

Example 2-Direct Fit Polynomials cont.

b) Change in Thermal Expansion Coefficient at -340^0F:
Since we want to find the rate of change in the thermal expansion coefficient at $T=-340^0F$, and we are using a third order polynomial, we need to choose the four points closest to $T=-340^0F$ that also bracket to $T=-340^0F$ evaluate it.

The four points are $T_0=-120\,°\text{F}$, $T_1=-200\,°\text{F}$, $T_2=-280\,°\text{F}$, and $T_3=-340\,°\text{F}$.

$$T_o=-120,\ \ \alpha(T_o)=5.09\times10^{-6}$$

$$T_1=-200,\ \ \alpha(T_1)=4.30\times10^{-6}$$

$$T_2=-280,\ \ \alpha(T_2)=3.33\times10^{-6}$$

$$T_3=-340,\ \ \alpha(T_3)=2.45\times10^{-6}$$

Example 2-Direct Fit Polynomials cont.

Such that

$$\alpha(-120) = 5.09 \times 10^{-6} = a_0 + a_1(-120) + a_2(-120)^2 + a_3(-120)^3$$

$$\alpha(-200) = 4.30 \times 10^{-6} = a_0 + a_1(-200) + a_2(-200)^2 + a_3(-200)^3$$

$$\alpha(-280) = 3.33 \times 10^{-6} = a_0 + a_1(-280) + a_2(-280)^2 + a_3(-280)^3$$

$$\alpha(-340) = 2.45 \times 10^{-6} = a_0 + a_1(-340) + a_2(-340)^2 + a_3(-340)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & -120 & 14400 & -1728000 \\ 1 & -200 & 40000 & -8000000 \\ 1 & -280 & 78400 & -21952000 \\ 1 & -340 & 115600 & -39304000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.09 \times 10^{-6} \\ 4.30 \times 10^{-6} \\ 3.33 \times 10^{-6} \\ 2.45 \times 10^{-6} \end{bmatrix}$$

Example 2-Direct Fit Polynomials cont.

Solving the above four equations gives

$$a_0 = 0.60625 \times 10^{-5}$$

$$a_1 = 0.74881 \times 10^{-8}$$

$$a_2 = -0.29018 \times 10^{-11}$$

$$a_3 = 0.18601 \times 10^{-13}$$

Hence

$$\begin{aligned}\alpha(T) &= a_0 + a_1 T + a_2 T^2 + a_3 T^3 \\ &= 0.60625 \times 10^{-5} + 0.74881 \times 10^{-8} T - 0.29018 \times 10^{-11} T^2 + 0.18601 \times 10^{-13} T^3, \\ &\qquad -340 \le T \le -120\end{aligned}$$

Example 2-Direct Fit Polynomials cont.

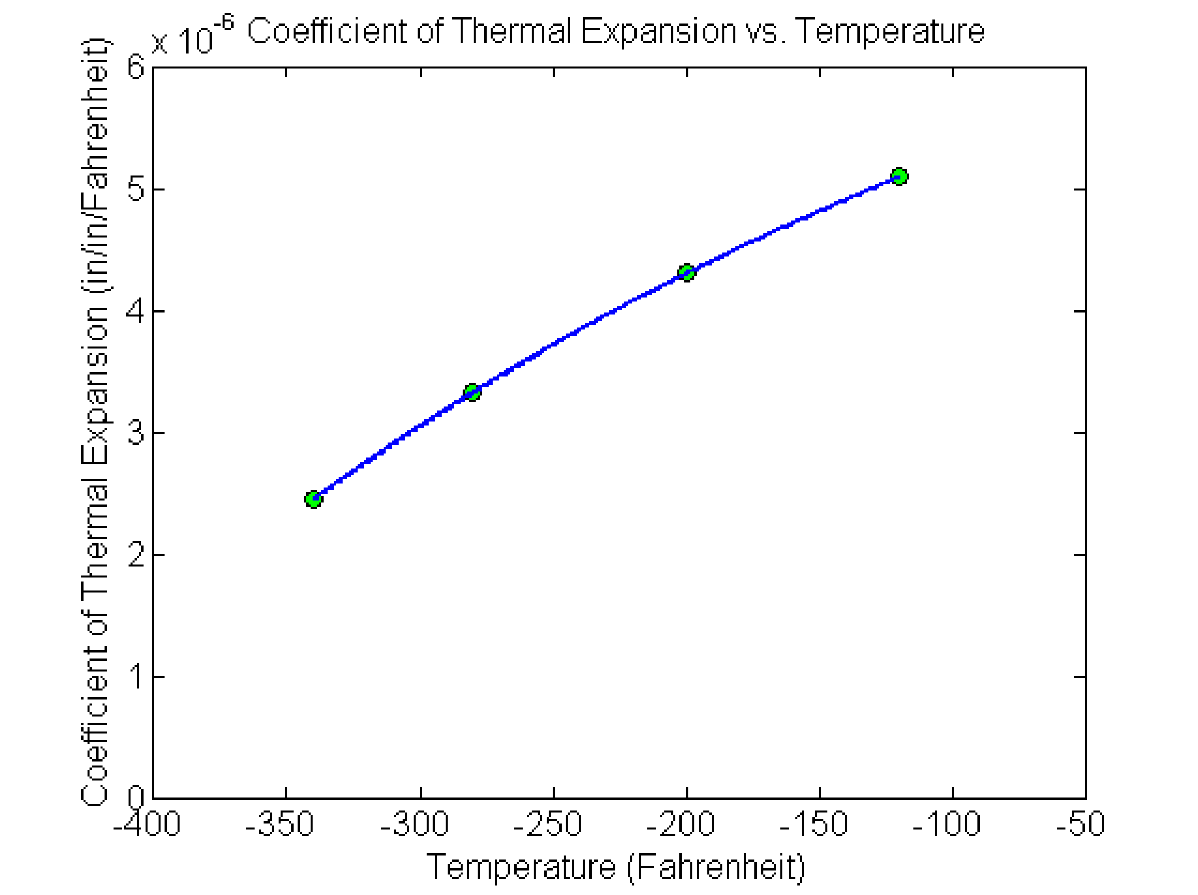

Figure 3 Graph of coefficient of thermal expansion vs. temperature.

Example 2-Direct Fit Polynomials cont.

The change in coefficient of thermal expansion at $T = -340^0 F$ is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dt}\alpha(T)\Big|_{T=-340}$$

Given that

$$\alpha(T) = 0.60625\times10^{-5} + 0.74881\times10^{-8}T - 0.29018\times10^{-11}T^2 + 0.18601\times10^{-13}T^3,$$
$$-340 \le T \le -120$$

$$\frac{d\alpha(T)}{dt} = \frac{d}{dt}\alpha(T)$$
$$= \frac{d}{dT}\left(0.60625\times10^{-5} + 0.74881\times10^{-8}T - 0.29018\times10^{-11}T^2 + 0.18601\times10^{-13}T^3\right)$$
$$= 0.74881\times10^{-8} - 0.58036\times10^{-11}T + 0.55804\times10^{-13}T^2,\ -340 \le t \le -120$$

Example 2-Direct Fit Polynomials cont.

$$\frac{d\alpha(-340)}{dt} = 0.74881\times10^{-8} - 0.58036\times10^{-11}(-340) + 0.55804\times10^{-13}(-340)^2$$

$$= 0.15905\times10^{-7}\ \text{in/in/°F}^2$$

Table 4 Summary of change in coefficient of thermal expansion using different approximations.

Temperature, T_i	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	3rd Order Interpolation	2nd Order Polynomial Regression
80^0F	$5.5563\times10^{-9}\ in/in/^oF^2$	$4.3246\times10^{-9}\ in/in/^oF^2$
-340^0F	$0.15905\times10^{-7}\ in/in/^oF^2$	$0.14585\times10^{-7}\ in/in/^oF^2$

Lagrange Polynomial

In this method, given $(x_1, y_1), \ldots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating equation (2) gives

Lagrange Polynomial Cont.

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Example 3

To find contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 5.

a) Using the second order Lagrange polynomial interpolant, find the change in coefficient of thermal expansion at $T = 80°\text{F}$ and $T = -340°\text{F}$.

b) The data given in Table 5 can be regressed to $\alpha = a_0 + a_1 T + a_2 T^2$ to get $\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}\text{T} - 1.2215\times10^{-11}\text{T}^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80°\text{F}$ and $T = -340°\text{F}$.

Example 3 Cont.

Table 5 Coefficient of thermal expansion as a function of temperature.

Temperature, T (°F)	Coefficient of thermal expansion, α (in/in/°F)
80	6.47×10^{-6}
40	6.24×10^{-6}
−40	5.72×10^{-6}
−120	5.09×10^{-6}
−200	4.30×10^{-6}
−280	3.33×10^{-6}
−340	2.45×10^{-6}

Example 3 Cont.

Solution

For second order Lagrangian interpolation, we choose the coefficient of thermal expansion given by

$$\alpha(T)=\left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\alpha(T_0)+\left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\alpha(T_1)+\left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\alpha(T_2)$$

a) Change in the thermal expansion coefficient at $80\,^\circ\mathrm{F}$:
Since we want to find the rate of change in the thermal expansion coefficient at $T=80\,^\circ\mathrm{F}$ and we are using second order Lagrangian interpolation, we need to choose the three points closest to $T=80\,^\circ\mathrm{F}$ that also bracket to $T=80\,^\circ\mathrm{F}$ evaluate it.

The three points are $T_0=80\,^\circ\mathrm{F},\ T_1=40\,^\circ\mathrm{F},\ \text{and}\ T_2=-40\,^\circ\mathrm{F}.$

$$T_0=80\,^\circ\mathrm{F},\ \alpha(T_0)=6.47\times10^{-6}$$
$$T_1=40\,^\circ\mathrm{F},\ \alpha(T_1)=6.24\times10^{-6}$$
$$T_2=-40\,^\circ\mathrm{F}\ \ \alpha(T_2)=5.72\times10^{-6}$$

Example 3 Cont.

The change in the coefficient of thermal expansion at $T = 80°\mathrm{F}$ is given by

$$\frac{d\alpha(80)}{dT} = \frac{d}{dT}\alpha(T)\Big|_{T=80}$$

Hence

$$\frac{d\alpha}{dT}(T) = \frac{2T-(T_1+T_2)}{(T_0-T_1)(T_0-T_2)}\alpha(T_0) + \frac{2T-(T_0+T_2)}{(T_1-T_0)(T_1-T_2)}\alpha(T_1) + \frac{2T-(T_0+T_1)}{(T_2-T_0)(T_2-T_1)}\alpha(T_2)$$

$$\frac{d\alpha}{dT}(80) = \frac{2(80)-(40+(-40))}{(80-40)(80-(-40))}(6.47\times10^{-6}) + \frac{2(80)-(80+(-40))}{(40-80)(40-(-40))}(6.24\times10^{-6})$$

$$+\frac{2(80)-(80+40)}{(-40-80)(-40-40)}(5.72\times10^{-6})$$

$$= 2.1567\times10^{-7} - 2.34\times10^{-7} + 2.3833\times10^{-8}$$

$$= 5.5\times10^{-9}\,\mathrm{in/in/°F^2}$$

Example 3 Cont.

b) Change in the thermal expansion coefficient at $-340\ ^\circ\text{F}$:
Since we want to find the rate of change in the thermal expansion coefficient at $T = -340\ ^\circ\text{F}$ and we are using second order Lagrangian interpolation, we need to choose the three points closest to $T = -340\ ^\circ\text{F}$ that also bracket to $T = -340\ ^\circ\text{F}$ evaluate it.

The three points are $T_0 = -200\ ^\circ\text{F},\ T_1 = -280\ ^\circ\text{F},$ and $T_2 = -340\ ^\circ\text{F}.$

$$T_0 = -200\ ^\circ\text{F},\ \alpha(T_0) = 4.30 \times 10^{-6}$$

$$T_1 = -280\ ^\circ\text{F},\ \alpha(T_1) = 3.33 \times 10^{-6}$$

$$T_2 = -340\ ^\circ\text{F}\ \ \alpha(T_2) = 2.45 \times 10^{-6}$$

The change in the coefficient of thermal expansion at $T = -340^\circ\text{F}$ is given by

$$\frac{d\alpha(-340)}{dT} = \frac{d}{dT}\alpha(T)\Big|_{T=-340}$$

Example 3 Cont.

Hence

$$\frac{d\alpha}{dT}(T)=\frac{2T-(T_1+T_2)}{(T_0-T_1)(T_0-T_2)}\alpha(T_0)+\frac{2T-(T_0+T_2)}{(T_1-T_0)(T_1-T_2)}\alpha(T_1)+\frac{2T-(T_0+T_1)}{(T_2-T_0)(T_2-T_1)}\alpha(T_2)$$

$$\frac{d\alpha}{dT}(-340)=\frac{2(-340)-(-280+(-340))}{(200-(-280))(-200-(-340))}\left(4.30\times10^{-6}\right)$$

$$+\frac{2(-340)-(-200+(-280))}{(-280-(-200))(-280-(-340))}\left(3.33\times10^{-6}\right)$$

$$+\frac{2(-340)-(-200+(-280))}{(-340-(-200))(-340-(-280))}\left(2.45\times10^{-6}\right)$$

$$=-2.3036\times10^{-8}+9.7125\times10^{-8}-5.8333\times10^{-8}$$

$$=0.15756\times10^{-7}\ \text{in/in/}^\circ\text{F}^2$$

Example 3 Cont.

Table 6 Summary of change in coefficient of thermal expansion using different approximations.

Temperature, T_i	Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$	
	2nd Order Lagrange Interpolation	2nd Order Polynomial Regression
80	5.5×10^{-9} in/in/°F^2	4.3246×10^{-9} in/in/°F^2
-340	0.15756×10^{-7} in/in/°F^2	0.14585×10^{-7} in/in/°F^2

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/discrete_02dif.html

THE END

http://numericalmethods.eng.usf.edu