Lagrangian Interpolation

Mechanical Engineering Majors

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Lagrange Method of Interpolation

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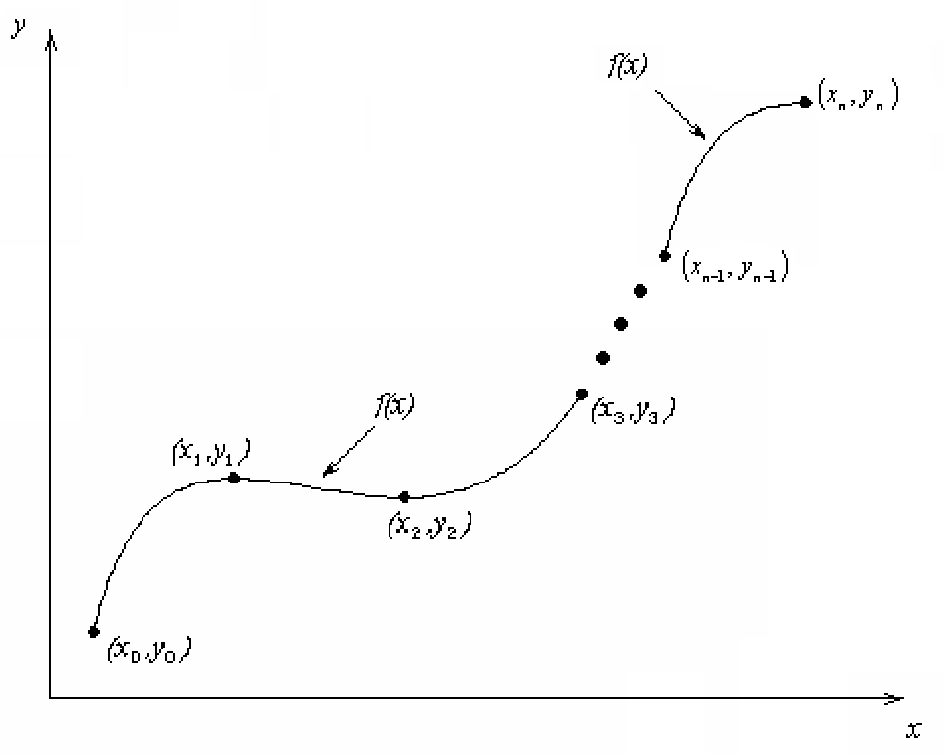
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$

given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using the Lagrangian method for linear interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
−60	5.58×10^{-6}
−160	4.72×10^{-6}
−260	3.58×10^{-6}
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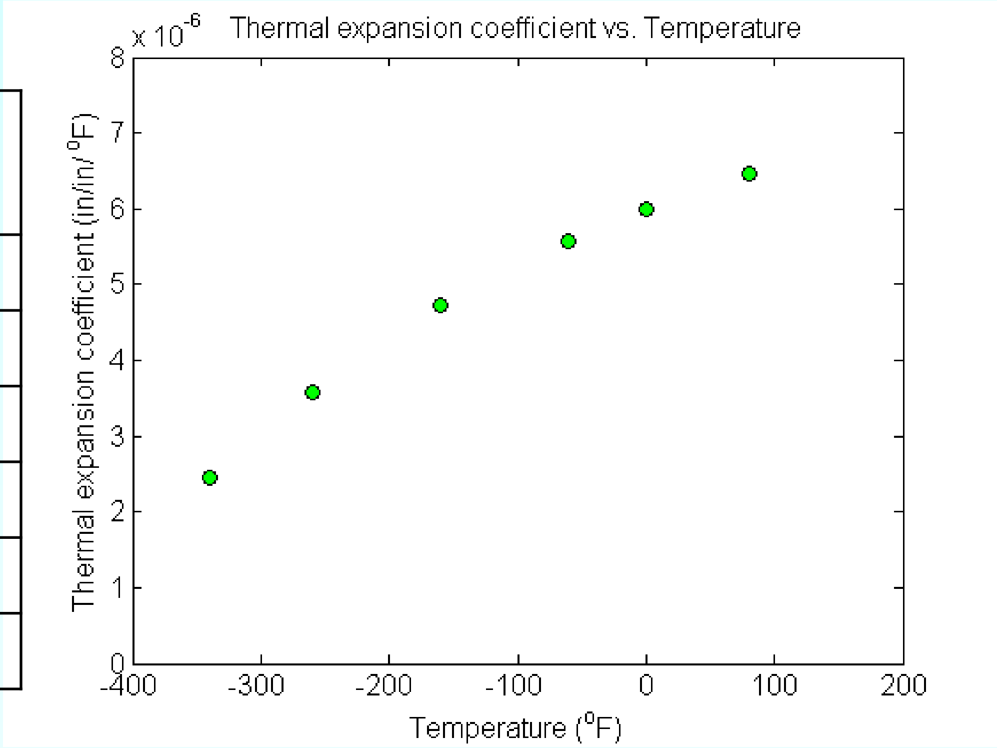
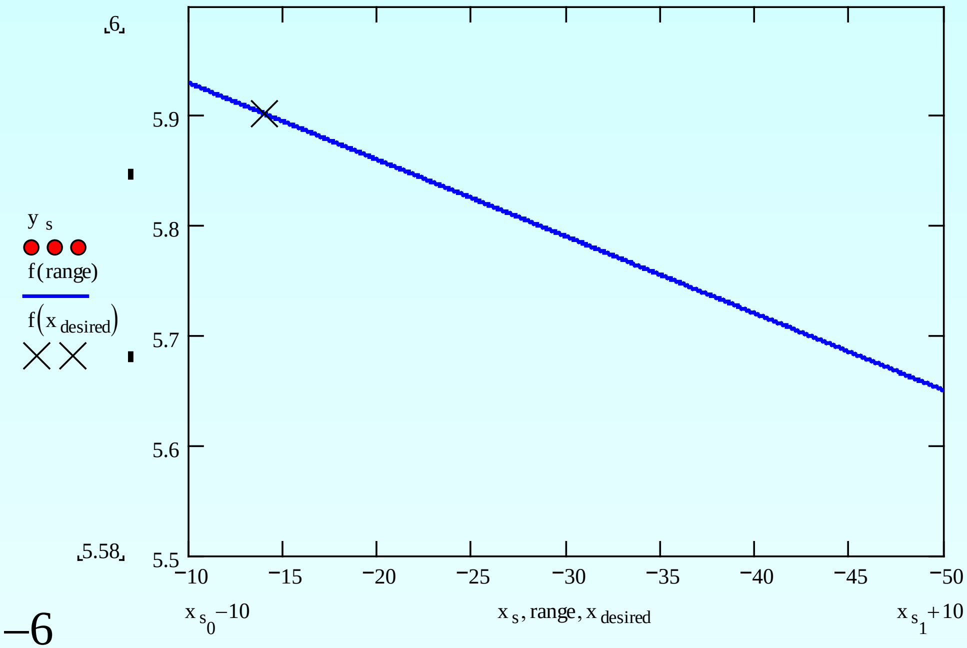


Linear Interpolation

$$\alpha(T) = \sum_{i=0}^{1} L_i(T)\alpha(T_i)$$

$$= L_0(T)\alpha(T_0) + L_1(T)\alpha(T_1)$$



$$T_0 = 0,\ \alpha(T_0) = 6.00\times 10^{-6}$$

$$T_1 = -60,\ \alpha(T_1) = 5.58\times 10^{-6}$$

Linear Interpolation (contd)

$$L_0(T)=\prod_{\substack{j=0\\j\neq 0}}^{1}\frac{T-T_j}{T_0-T_j}=\frac{T-T_1}{T_0-T_1}$$

$$L_1(T)=\prod_{\substack{j=0\\j\neq 1}}^{1}\frac{T-T_j}{T_1-T_j}=\frac{T-T_0}{T_1-T_0}$$

$$\alpha(T)=\frac{T-T_1}{T_0-T_1}\alpha(T_0)+\frac{T-T_0}{T_1-T_0}\alpha(T_1)$$

$$=\frac{T+60}{0+60}\left(6.00\times10^{-6}\right)+\frac{T-0}{-60-0}\left(5.58\times10^{-6}\right),\ -60\le T\le 0$$

$$\alpha(-14)=\frac{-14+60}{0+60}\left(6.00\times10^{-6}\right)+\frac{-14-0}{-60-0}\left(5.58\times10^{-6}\right)$$

$$=0.76667\left(6.00\times10^{-6}\right)+0.23333\left(5.58\times10^{-6}\right)$$

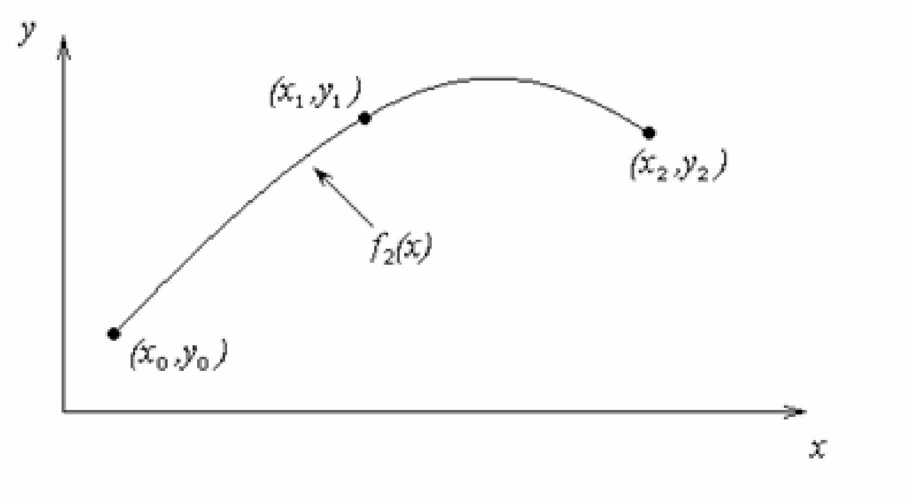
$$=5.902\times10^{-6}\ \text{in/in/°F}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using the Lagrangian method for quadratic interpolation.

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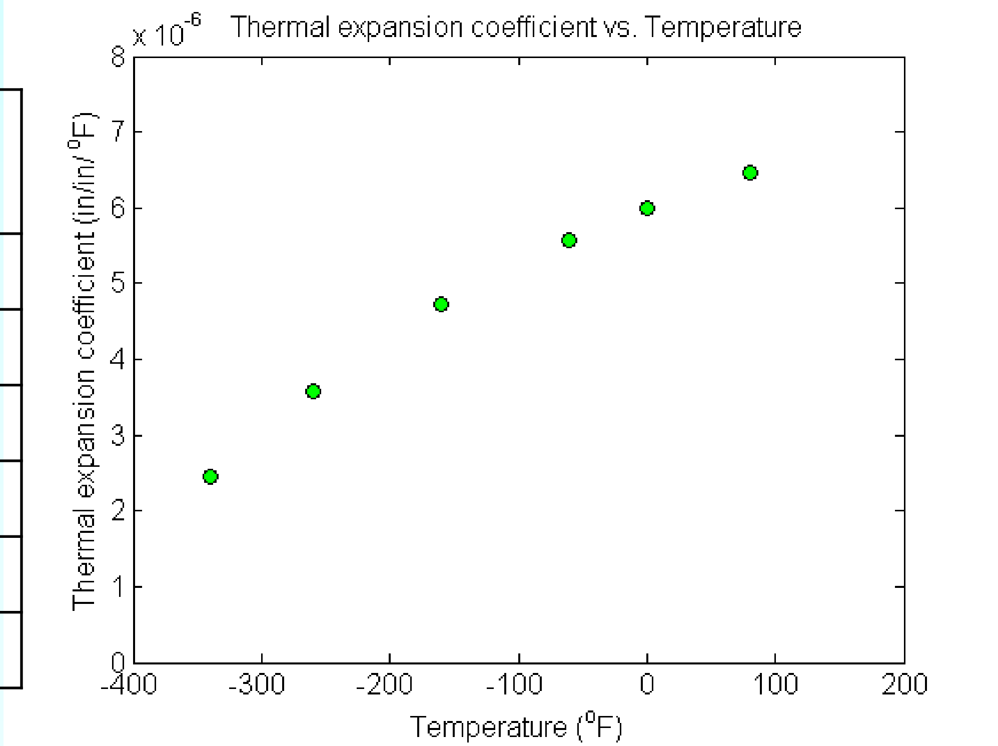


Quadratic Interpolation (contd)

$$T_o = 80,\ \alpha(T_o) = 6.47\times10^{-6}$$

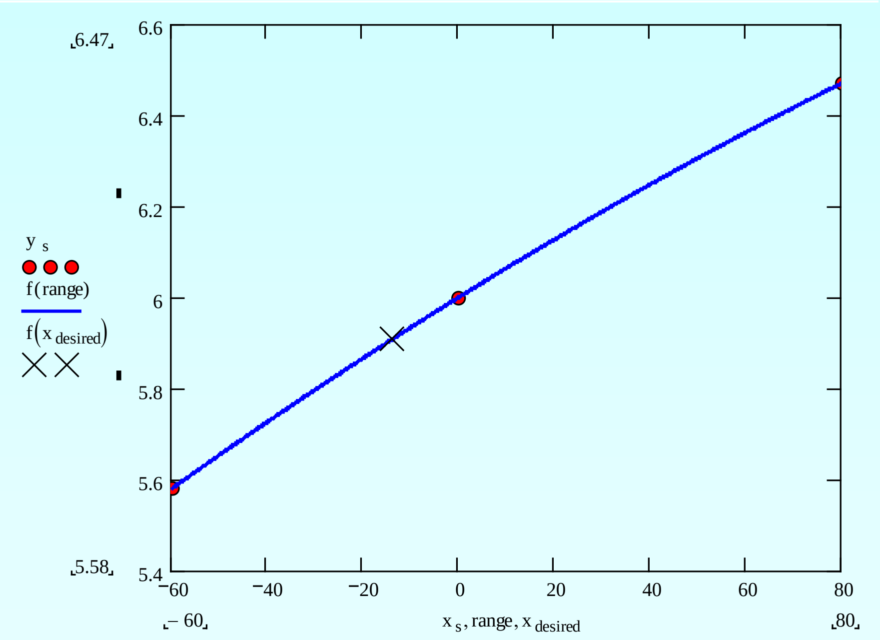
$$T_1 = 0,\ \alpha(T_1) = 6.00\times10^{-6}$$

$$T_2 = -60,\ \alpha(T_2) = 5.58\times10^{-6}$$

$$L_0(T) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{T-T_j}{T_0-T_j} = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{T-T_j}{T_1-T_j} = \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{T-T_j}{T_2-T_j} = \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)$$



Quadratic Interpolation (contd)

$$\alpha(T) = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\alpha(T_0) + \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\alpha(T_1) + \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\alpha(T_2)$$

$$\alpha(-14) = \frac{(-14-0)(-14+60)}{(80-0)(80+60)}\left(6.47\times10^{-6}\right) + \frac{(-14-80)(-14+60)}{(0-80)(0+60)}\left(6.00\times10^{-6}\right)$$

$$+ \frac{(-14-80)(-14-0)}{(-60-80)(-60-0)}\left(5.58\times10^{-6}\right)$$

$$= (-0.0575)\left(6.47\times10^{-6}\right) + (0.90083)\left(6.00\times10^{-6}\right) + (0.15667)\left(5.58\times10^{-6}\right)$$

$$= 5.9072\times10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{5.9072\times10^{-6} - 5.902\times10^{-6}}{5.9072\times10^{-6}}\right| \times 100$$

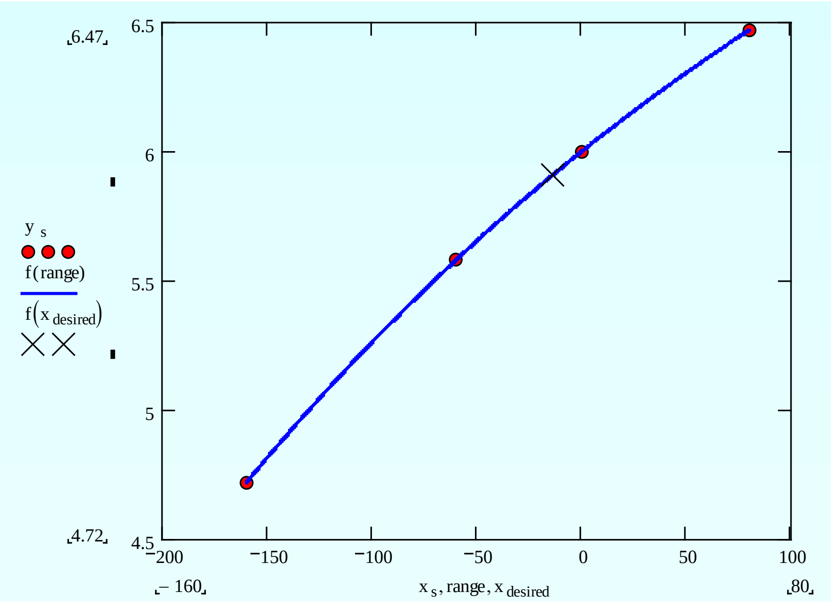
$$= 0.087605\%$$

Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T) = \sum_{i=0}^{3} L_i(T)\alpha(T_i)$$

$$= L_0(T)\alpha(T_0) + L_1(T)\alpha(T_1) + L_2(T)\alpha(T_2) + L_3(T)\alpha(T_3)$$



Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using the Lagrangian method for cubic interpolation.

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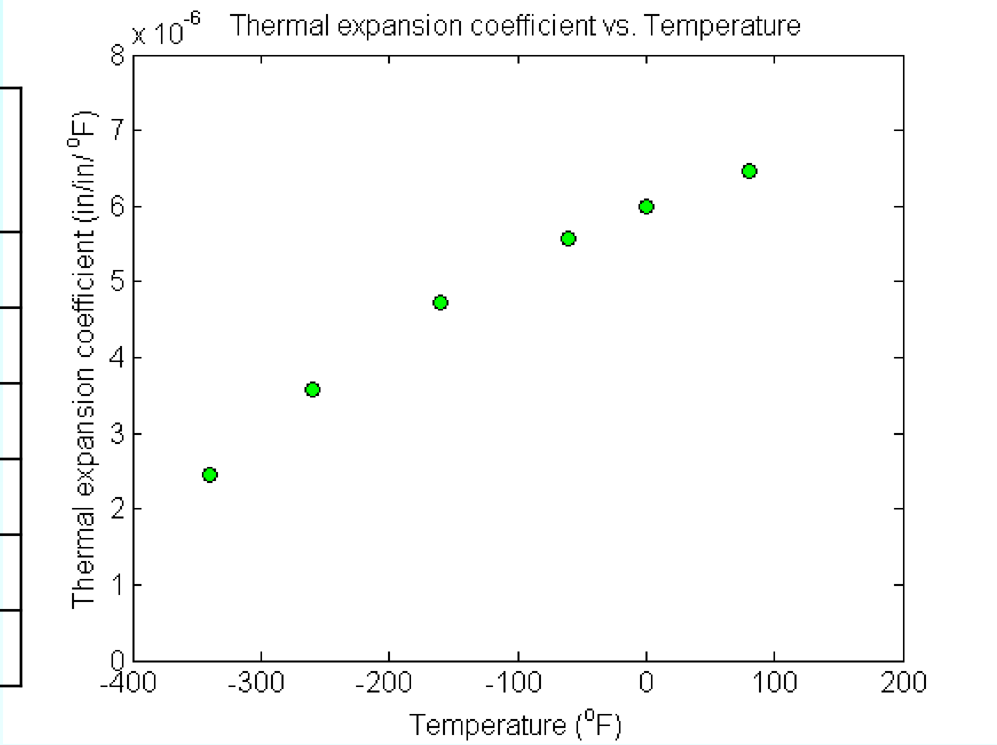


Cubic Interpolation (contd)

$$T_o = 80,\ \alpha(T_o) = 6.47\times10^{-6} \qquad T_1 = 0,\ \alpha(T_1) = 6.00\times10^{-6}$$

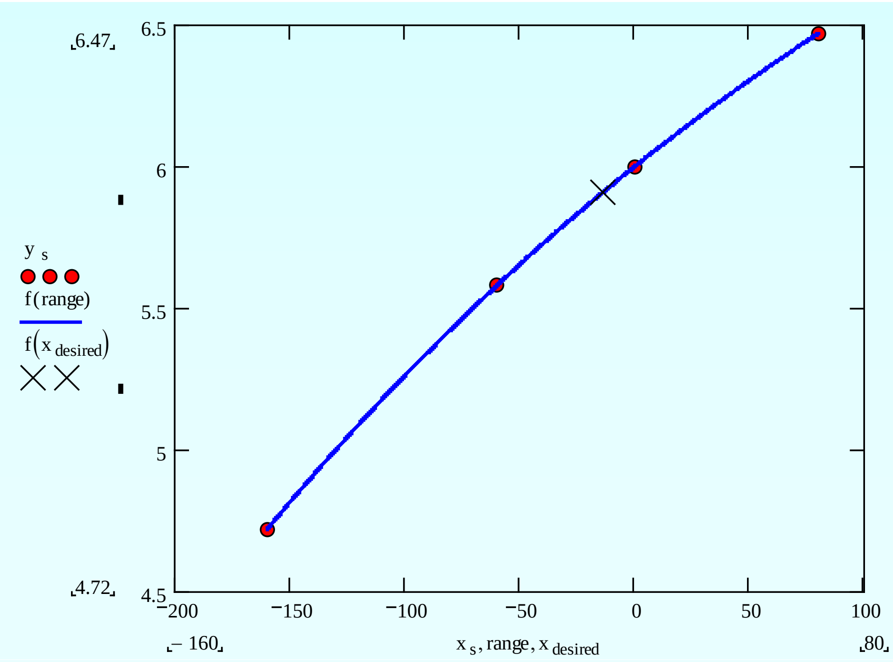
$$T_2 = -60,\ \alpha(T_2) = 5.58\times10^{-6} \qquad T_3 = -160,\ \alpha(T_3) = 4.72\times10^{-6}$$

$$L_0(T) = \prod_{\substack{j=0\\ j\neq 0}}^{3} \frac{T-T_j}{T_0-T_j} = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\left(\frac{T-T_3}{T_0-T_3}\right)$$

$$L_1(T) = \prod_{\substack{j=0\\ j\neq 1}}^{3} \frac{T-T_j}{T_1-T_j} = \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\left(\frac{T-T_3}{T_1-T_3}\right)$$

$$L_2(T) = \prod_{\substack{j=0\\ j\neq 2}}^{3} \frac{T-T_j}{T_2-T_j} = \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\left(\frac{T-T_3}{T_2-T_3}\right)$$

$$L_3(T) = \prod_{\substack{j=0\\ j\neq 3}}^{3} \frac{T-T_j}{T_3-T_j} = \left(\frac{T-T_0}{T_3-T_0}\right)\left(\frac{T-T_1}{T_3-T_1}\right)\left(\frac{T-T_2}{T_3-T_2}\right)$$



Cubic Interpolation (contd)

$$\alpha(T)=\left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\left(\frac{T-T_3}{T_0-T_3}\right)\alpha(T_0)+\left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\left(\frac{T-T_3}{T_1-T_3}\right)\alpha(T_1)$$

$$+\left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\left(\frac{T-T_3}{T_2-T_3}\right)\alpha(T_0)+\left(\frac{T-T_0}{T_3-T_0}\right)\left(\frac{T-T_1}{T_3-T_1}\right)\left(\frac{T-T_2}{T_3-T_2}\right)\alpha(T_1), \quad T_0 \le T \le T_3$$

$$\alpha(-14)=\frac{(-14-0)(-14+60)(-14+160)}{(80-0)(80+60)(80+160)}\left(6.47\times10^{-6}\right)+\frac{(-14-80)(-14+60)(-14+160)}{(0-80)(0+60)(0+160)}\left(6.00\times10^{-6}\right)$$

$$+\frac{(-14-80)(-14-0)(-14+160)}{(-60-80)(-60-0)(-60+160)}\left(5.58\times 10^{-6}\right)+\frac{(-14-80)(-14-0)(-14+60)}{(-160-80)(-160-0)(-160+60)}\left(4.72\times10^{-6}\right)$$

$$=(-0.034979)\left(6.47\times10^{-6}\right)+(0.82201)\left(6.00\times10^{-6}\right)+(0.22873)\left(5.58\times10^{-6}\right)+(-0.015765)\left(4.72\times10^{-6}\right)$$

$$=5.9077\times10^{-6}\ \text{in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a|=\left|\frac{5.9077\times10^{-6}-5.9072\times10^{-6}}{5.9077\times10^{-6}}\right|\times100$$

$$=0.0083867\%$$

Comparison Table

Order of Polynomial	1	2	3
Thermal Expansion Coefficient (in/in/^{o}F)	5.902×10^{-6}	5.9072×10^{-6}	5.9077×10^{-6}
Absolute Relative Approximate Error	----------	0.087605%	0.0083867%

Reduction in Diameter

The actual reduction in diameter is given by

$$\Delta D = D\int_{Tr}^{Tf} \alpha dT$$

where T_r = room temperature (°F)

T_f = temperature of cooling medium (°F)

Since T_r = 80 °F and T_r = −108 °F, $\Delta D = D\int_{80}^{-108} \alpha dT$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from the cubic interpolation.

Reduction in Diameter

We know from interpolation that

$$\alpha(T) = 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1944\times10^{-12}T^2 + 8.1845\times10^{-15}T^3,$$

$$-160 \le T \le 80$$

Therefore,

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha\, dT$$

$$= \int_{80}^{-108} \left(6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1944\times10^{-12}T^2 + 8.1845\times10^{-15}T^3\right) dT$$

$$= \left[6.00\times10^{-6}T + 6.4786\times10^{-9}\frac{T^2}{2} - 8.1944\times10^{-12}\frac{T^3}{3} + 8.1845\times10^{-15}\frac{T^4}{4}\right]_{80}^{-180}$$

$$= -1105.9\times10^{-6}$$

Reduction in diameter

Using the average value for the coefficient of thermal expansion from cubic interpolation

$$\frac{\Delta D}{D} = \alpha \Delta T$$

$$= \alpha \left(T_f - T_r \right)$$

$$= 5.9077 \times 10^{-6} \left(-108 - 80 \right)$$

$$= -1110.6 \times 10^{-6}$$

The percentage difference would be

$$\left| \in_a \right| = \left| \frac{-1105.9 \times 10^{-6} - \left(-1110.6 \times 10^{-6} \right)}{-1105.9 \times 10^{-6}} \right| \times 100$$

$$= 0.42775\%$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lagrange_method.html

THE END

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