Newton's Divided Difference Polynomial Method of Interpolation

Mechanical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

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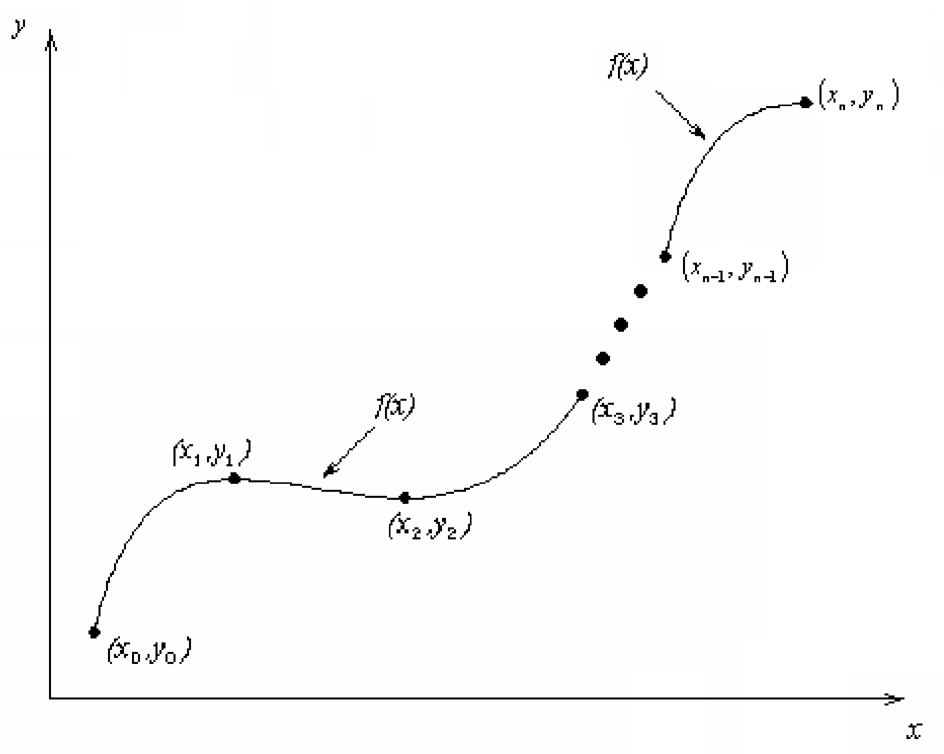
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

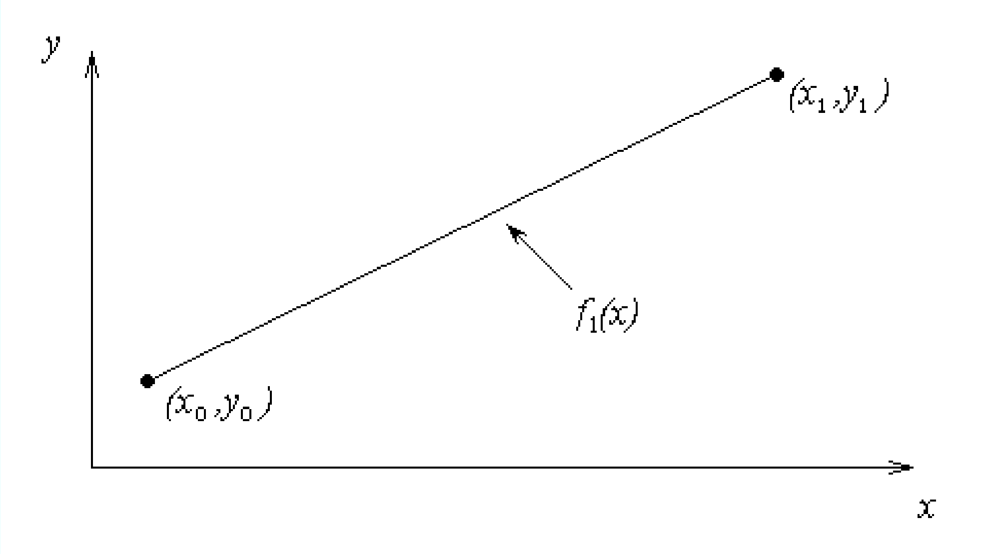
Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using the direct method for linear interpolation.

Temperature (°F)	Thermal Expansion Coefficient (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
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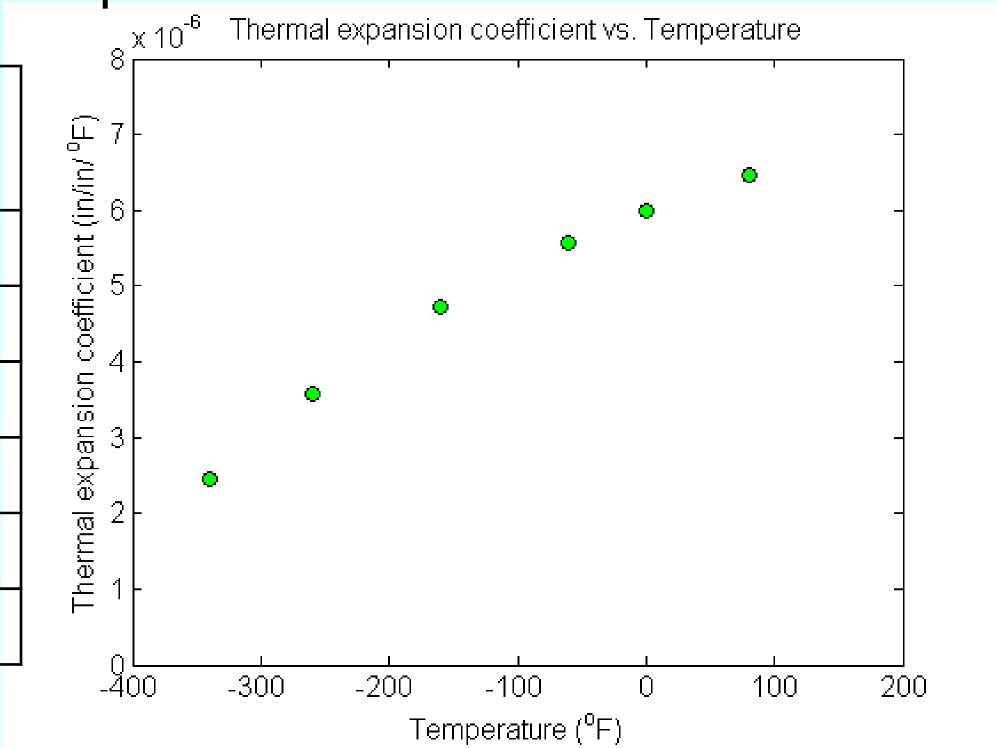


Linear Interpolation

$$\alpha(T) = b_0 + b_1(T - T_0)$$

$$T_0 = 0,\ \alpha(T_0) = 6.00\times10^{-6}$$

$$T_1 = -60,\ \alpha(T_1) = 5.58\times10^{-6}$$

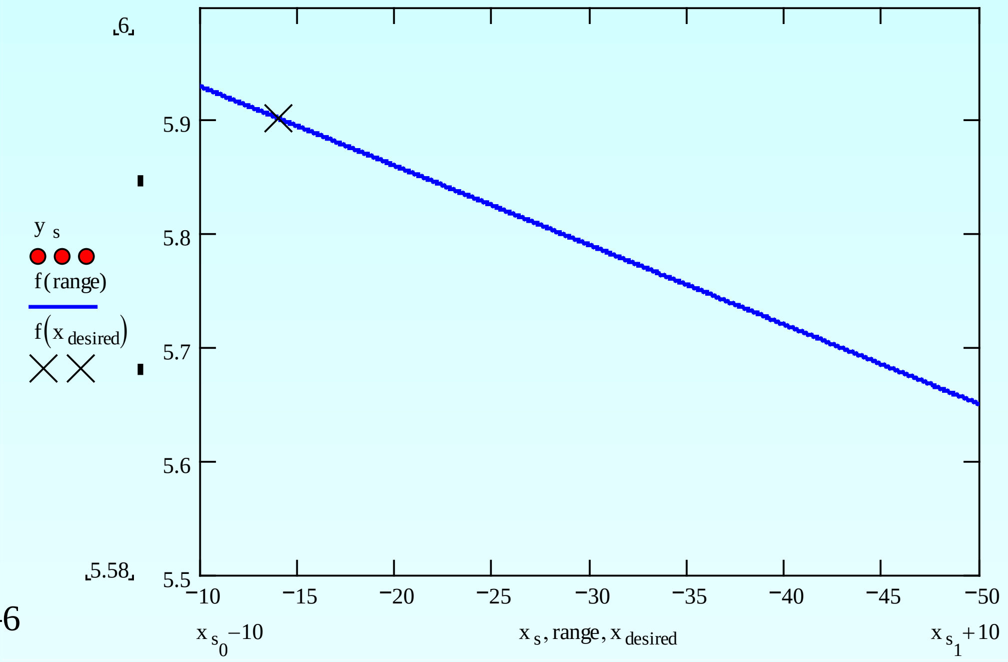
$$b_0 = \alpha(T_0)$$

$$= 6.00\times10^{-6}$$

$$b_1 = \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0}$$

$$= \frac{5.58\times10^{-6} - 6.00\times10^{-6}}{-60 - 0}$$

$$= 0.007\times10^{-6}$$



Linear Interpolation (contd)

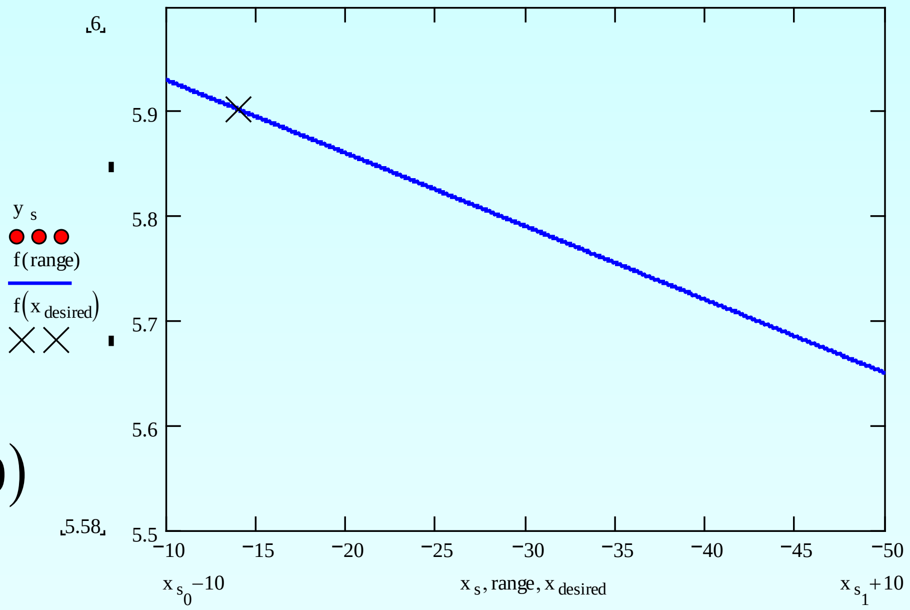
$$\alpha(T) = b_0 + b_1(T - T_0)$$

$$= 6.00\times10^{-6} + 0.007\times10^{-6}(T - 0),$$

$$-60 \le T \le 0$$

$$\alpha(-14) = 6.00\times10^{-6} + 0.007\times10^{-6}(-14 - 0)$$

$$= 5.902\times10^{-6}\ \text{in/in/°F}$$



Quadratic Interpolation

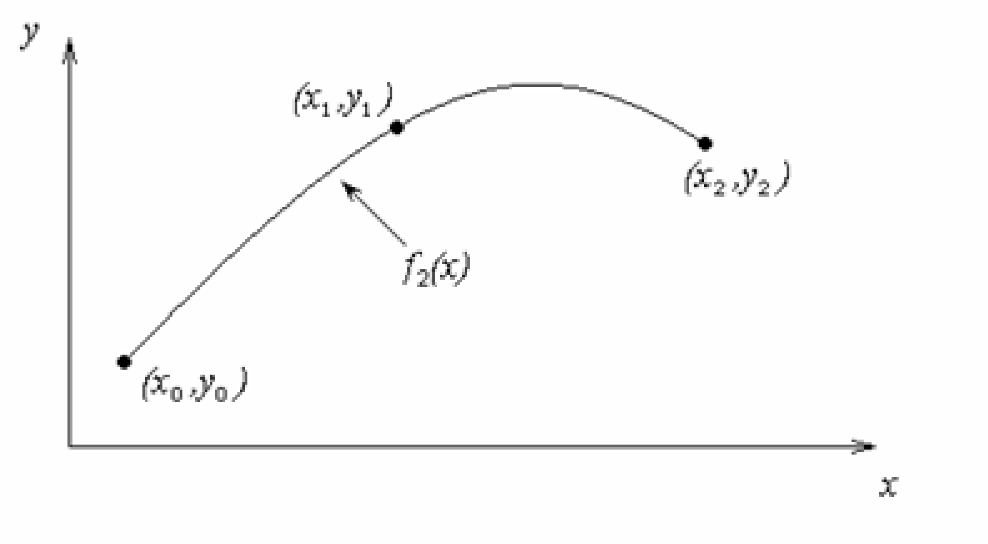
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using the direct method for quadratic interpolation.

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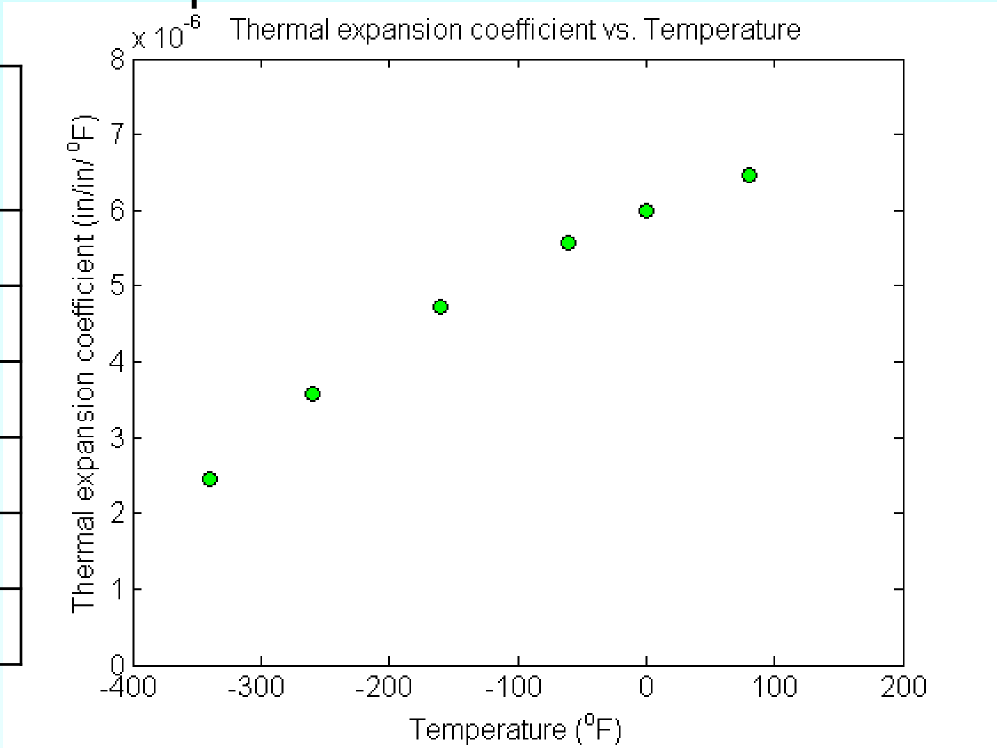


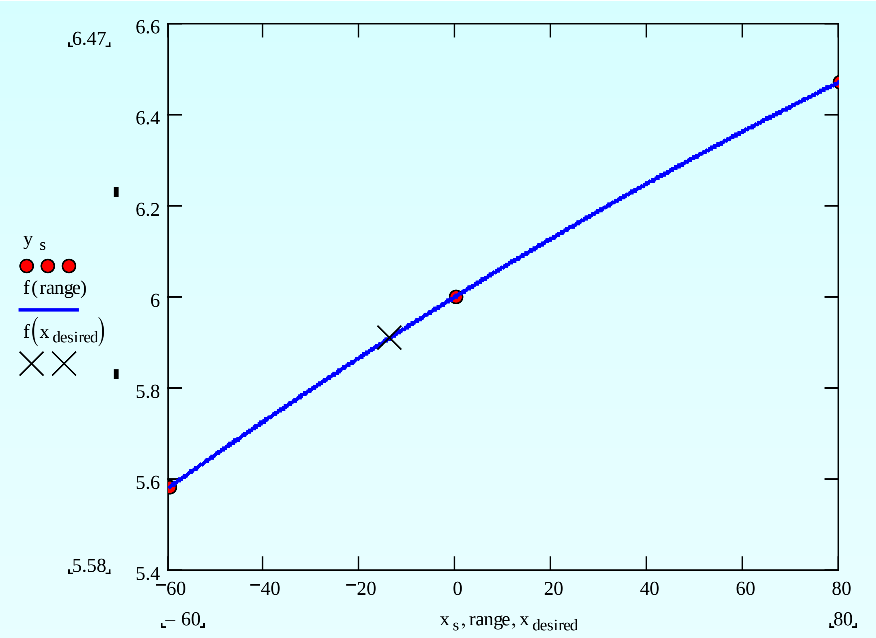
Quadratic Interpolation (contd)

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$T_0 = 80, \qquad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 0, \qquad \alpha(T_1) = 6.00 \times 10^{-6}$$

$$T_2 = -60, \qquad \alpha(T_2) = 5.58 \times 10^{-6}$$



Quadratic Interpolation (contd)

$$b_0 = \alpha(T_0) = 6.47\times10^{-6}$$

$$b_1 = \frac{\alpha(T_1)-\alpha(T_0)}{T_1 - T_0} = \frac{6.00\times10^{-6} - 6.47\times10^{-6}}{0-80} = 5.875\times10^{-9}$$

$$b_2 = \frac{\dfrac{\alpha(T_2)-\alpha(T_1)}{T_2-T_1} - \dfrac{\alpha(T_1)-\alpha(T_0)}{T_1-T_0}}{T_2-T_0}$$

$$= \frac{\dfrac{5.58\times10^{-6} - 6.00\times10^{-6}}{-60-0} - \dfrac{6.00\times10^{-6} - 6.47\times10^{-6}}{0-80}}{-60-80}$$

$$= \frac{0.007\times10^{-6} - 0.005875\times10^{-6}}{-140}$$

$$= -8.0357\times10^{-12}$$

Quadratic Interpolation (contd)

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$= 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(T - 80) - 8.0357 \times 10^{-12}(T - 80)(T - 0), \; -60 \le T \le 80$$

At $T = -14$,

$$\alpha(-14) = 6.47 \times 10^{-6} + 5.875 \times 10^{-9}(-14 - 80) - 8.0357 \times 10^{-12}(-14 - 80)(-14 - 0)$$

$$= 5.9072 \times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100$$

$$= 0.087605\%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

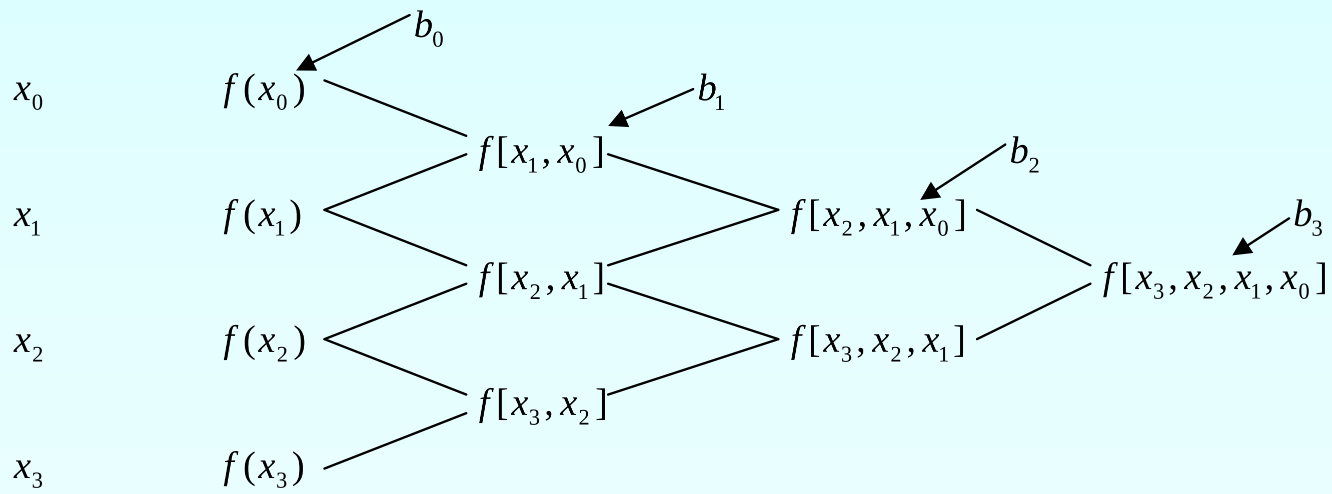
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

A trunnion is cooled 80°F to − 108°F. Given below is the table of the coefficient of thermal expansion vs. temperature. Determine the value of the coefficient of thermal expansion at T=−14°F using the direct method for cubic interpolation.

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80	6.47×10^{-6}
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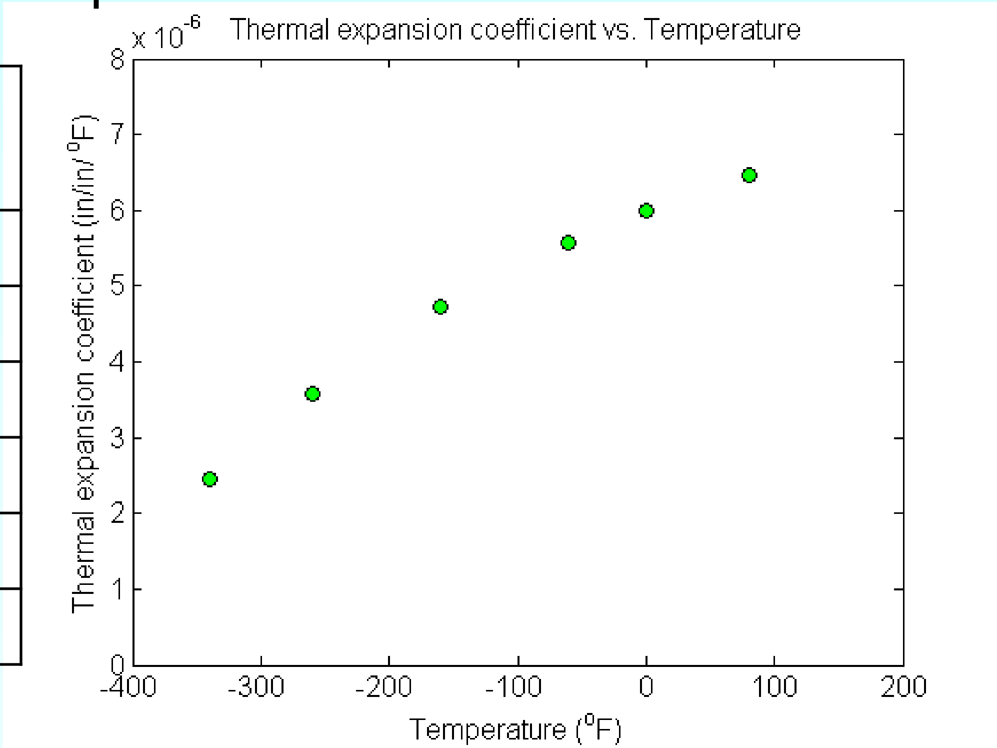

Example

The coefficient of thermal expansion profile is chosen as

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

$$T_0 = 80, \qquad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 0, \qquad \alpha(T_1) = 6.00 \times 10^{-6}$$

$$T_2 = -60, \qquad \alpha(T_2) = 5.58 \times 10^{-6}$$

$$T_3 = -160, \qquad \alpha(T_3) = 4.72 \times 10^{-6}$$

Example

$T_0 = 80,$	6.47×10^{-6} (b_0)			
		5.875×10^{-9} (b_1)		
$T_1 = 0,$	6.00×10^{-6}		-8.0357×10^{-12} (b_2)	
		0.007×10^{-6}		8.1845×10^{-15} (b_3)
$T_2 = -60,$	5.58×10^{-6}		-10^{-11}	
		0.0086×10^{-6}		
$T_3 = -160,$	4.72×10^{-6}			

The values of the constants are

$$b_0 = 6.47\times10^{-6} \quad b_1 = 5.875\times10^{-9} \quad b_2 = -8.0357\times10^{-12} \quad b_3 = 8.1845\times10^{-15}$$

Example

$$\alpha(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

$$= 6.47\times10^{-6} + 5.875\times10^{-9}(T - 80) - 8.0357\times10^{-12}(T - 80)(T - 0)$$

$$+ 8.1845\times10^{-15}(T\text{-}80)(T\text{-}0)(T + 60)$$

At $T = -14$,

$$\alpha(-14) = 6.47\times10^{-6} + 5.875\times10^{-9}(-14 - 80) - 8.0357\times10^{-12}(-14 - 80)(-14 - 0)$$

$$+ 8.1845\times10^{-15}(-14 - 80)(-14 - 0)(-14 + 60)$$

$$= 5.9077\times10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{5.9077\times10^{-6} - 5.9072\times10^{-6}}{5.9077\times10^{-6}}\right| \times 100$$

$$= 0.0083867\%$$

Comparison Table

Order of Polynomial	1	2	3
Thermal Expansion Coefficient (in/in/^{o}F)	5.902×10^{-6}	5.9072×10^{-6}	5.9077×10^{-6}
Absolute Relative Approximate Error	----------	0.087605%	0.0083867%

Reduction in Diameter

The actual reduction in diameter is given by

$$\Delta D = D\int_{Tr}^{Tf} \alpha dT$$

where T_r = room temperature (°F)

T_f = temperature of cooling medium (°F)

Since T_r = 80 °F and T_r = −108 °F, $\Delta D = D\int_{80}^{-108} \alpha dT$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from the cubic interpolation.

Reduction in Diameter

We know from interpolation that

$$\alpha(T) = 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1944\times10^{-12}T^2 + 8.1845\times10^{-15}T^3,$$

$$-160 \le T \le 80$$

Therefore,

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha\, dT$$

$$= \int_{80}^{-108} \left(6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1944\times10^{-12}T^2 + 8.1845\times10^{-15}T^3\right) dT$$

$$= \left[6.00\times10^{-6}T + 6.4786\times10^{-9}\frac{T^2}{2} - 8.1944\times10^{-12}\frac{T^3}{3} + 8.1845\times10^{-15}\frac{T^4}{4}\right]_{80}^{-180}$$

$$= -1105.9\times10^{-6}$$

Reduction in diameter

Using the average value for the coefficient of thermal expansion from cubic interpolation

$$\frac{\Delta D}{D} = \alpha \Delta T$$

$$= \alpha\left(T_f - T_r\right)$$

$$= 5.9077 \times 10^{-6}\left(-108 - 80\right)$$

$$= -1110.6 \times 10^{-6}$$

The percentage difference would be

$$\left|\in_a\right| = \left|\frac{-1105.9 \times 10^{-6} - \left(-1110.6 \times 10^{-6}\right)}{-1105.9 \times 10^{-6}}\right| \times 100$$

$$= 0.42775\%$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html

THE END

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