# Runge 2nd Order Method

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# Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x,y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

# Heun's Method

<u>Heun's method</u>

Here $a_2$=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + h, y_i + k_1 h\right)$$

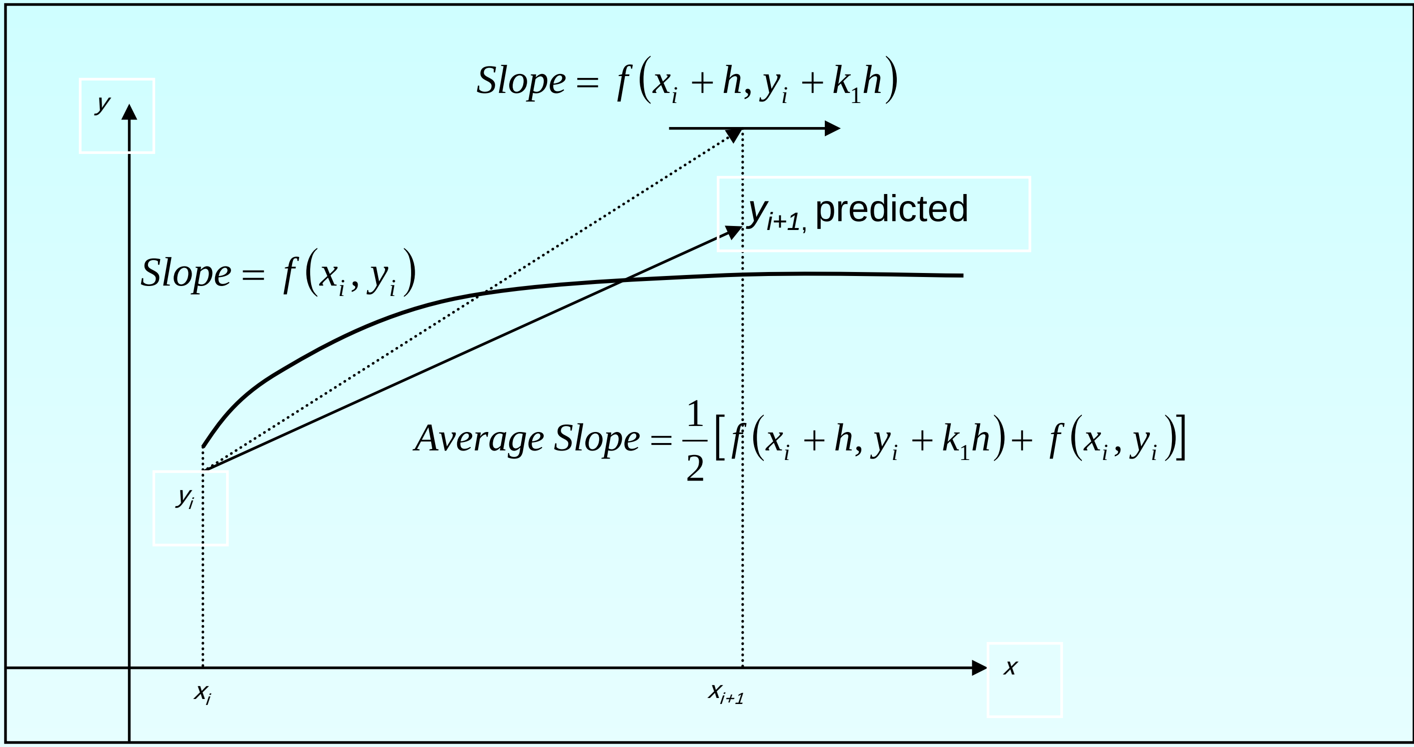


**Figure 1** Runge-Kutta 2nd order method (Heun's method)

# Midpoint Method

**Here** $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

# Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

# How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

# Example

A solid steel shaft at room temperature of 27°C is needed to be contracted so it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at −33°C. The rate of change of temperature of the solid shaft $\theta$ is given by

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588\end{pmatrix}(\theta+33)$$

$$\theta(0) = 27°\text{C}$$

Find the temperature of the steel shaft after 24 hours. Take a step size of $h$ = 43200 seconds.

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\left(-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.588\right)(\theta+33)$$

$$f(t,\theta) = -5.33\times10^{-6}\left(-3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + 5.42\times10^{-2}\theta + 5.588\right)(\theta+33)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

# Solution

Step 1: For $i = 0,\ t_0 = 0,\ \theta_0 = 27$

$$k_1 = f(t_0,\ \theta_o) = f(0,\ 27) = \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(27)^4 + 2.33\times10^{-5}(27)^3 \\ +1.35\times10^{-3}(27)^2 + 5.42\times10^{-2}(27) + 5.588\end{pmatrix}(27+33)\right)$$

$$= -0.0020893$$

$$k_2 = f(t_0 + h,\ \theta_0 + k_1 h) = f(0 + 43200,\ 27 + (-0.0020893)43200) = f(43200, -63.278)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-63.278)^4 + 2.33\times10^{-5}(-63.278)^3 \\ +1.35\times10^{-3}(-63.278)^2 + 5.42\times10^{-2}(-63.278) + 5.588\end{pmatrix}(-63.278+33)\right)$$

$$= -0.0092607$$

$$\theta_1 = \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h = 27 + \left(\frac{1}{2}(-0.0020893) + \frac{1}{2}(-0.0092607)\right)43200$$

$$= 27 + (-0.0056750)43200 = -218.16^\circ C$$

# Solution Cont

**Step 2:** $i = 1,\ t_1 = 43200,\ \theta_1 = -218.16°C$

$$k_1 = f\left(t_1, \theta_1\right) = f\left(43200, -218.16\right)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}\left(-218.16\right)^4 + 2.33\times10^{-5}\left(-218.16\right)^3 \\ +1.35\times10^{-3}\left(-218.16\right)^2 + 5.42\times10^{-2}\left(-218.16\right) + 5.588\end{pmatrix}\left(-218.16+33\right)\right)$$

$$= -8.4304$$

$$k_2 = f\left(t_1 + h,\ \theta_1 + k_1 h\right) = f\left(43200 + 43200, -218.16 + \left(-8.4304\right)43200\right) = f\left(86400, -364410\right)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-364410)^4 + 2.33\times10^{-5}(-364410)^3 \\ +1.35\times10^{-3}\left(-364410\right)^2 + 5.42\times10^{-2}\left(-364410\right) + 5.588\end{pmatrix}\left(-364410+33\right)\right)$$

$$= -1.2638\times10^{17}$$

$$\theta_2 = \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h = -218.16 + \left(\frac{1}{2}\left(-8.4304\right) + \frac{1}{2}\left(-1.2638\times10^{17}\right)\right)43200$$

$$= -218.16 + \left(-6.3190\times10^{16}\right)43200 = -2.7298\times10^{21}°\text{C}$$

# Solution Cont

The solution to this nonlinear equation at *t=86400s* is

$$\theta(86400) = -26.099°C$$

# Comparison with exact results

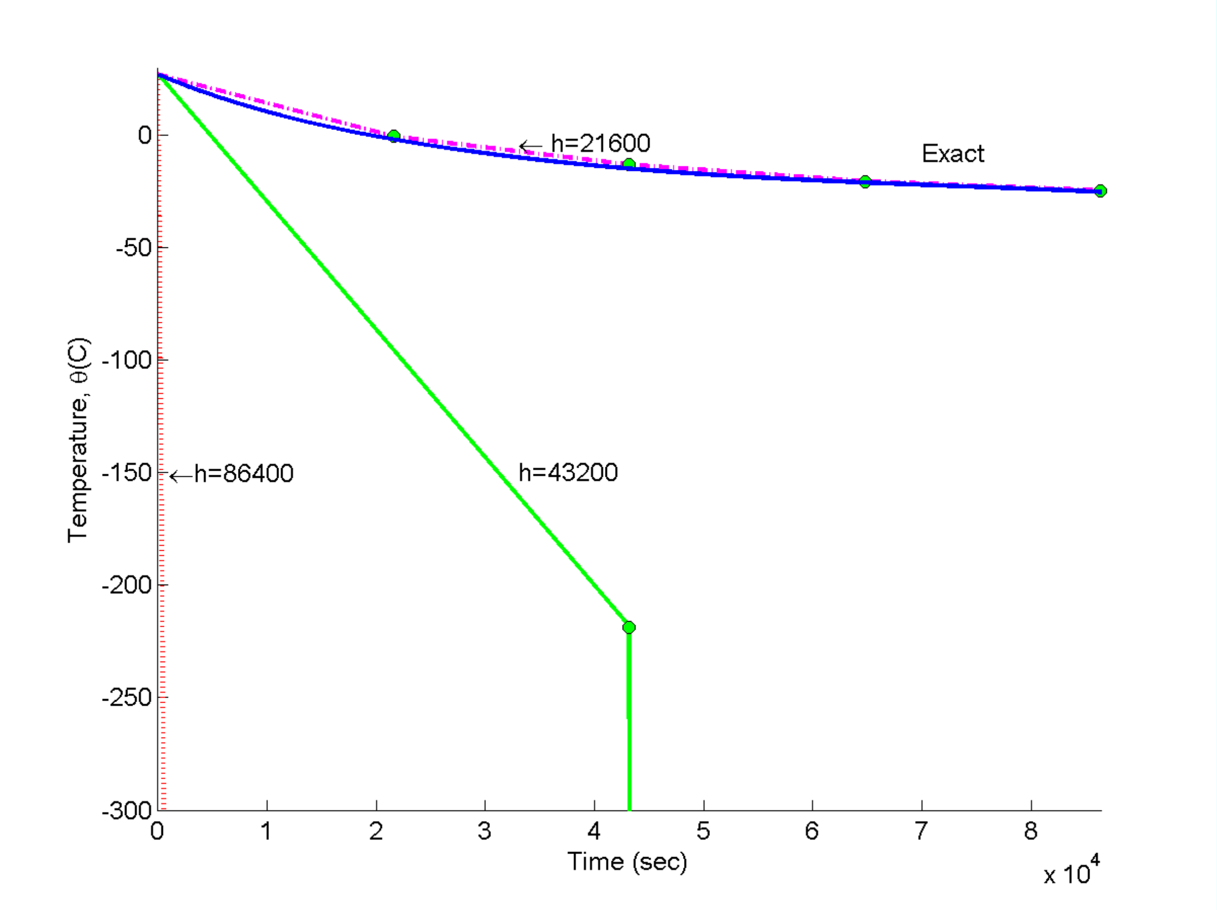


**Figure 2.** Heun's method results for different step sizes

# Effect of step size

**Table 1.** Effect of step size for Heun's method

| Step size, $h$ | $\theta(86400)$ | $E_t$ | $\lvert \in_t \rvert \%$ |
|---|---|---|---|
| 86400 | $-58466$ | 58440 | 223920 |
| 43200 | $-2.7298 \times 10^{21}$ | $2.7298 \times 10^{21}$ | $1.0460 \times 10^{11}$ |
| 21600 | $-24.537$ | $-1.5619$ | 5.9845 |
| 10800 | $-25.785$ | $-0.31368$ | 1.2019 |
| 5400 | $-26.027$ | $-0.072214$ | 0.27670 |

$\theta(86400) = -26.099°C$ (exact)

# Effects of step size on Heun's Method

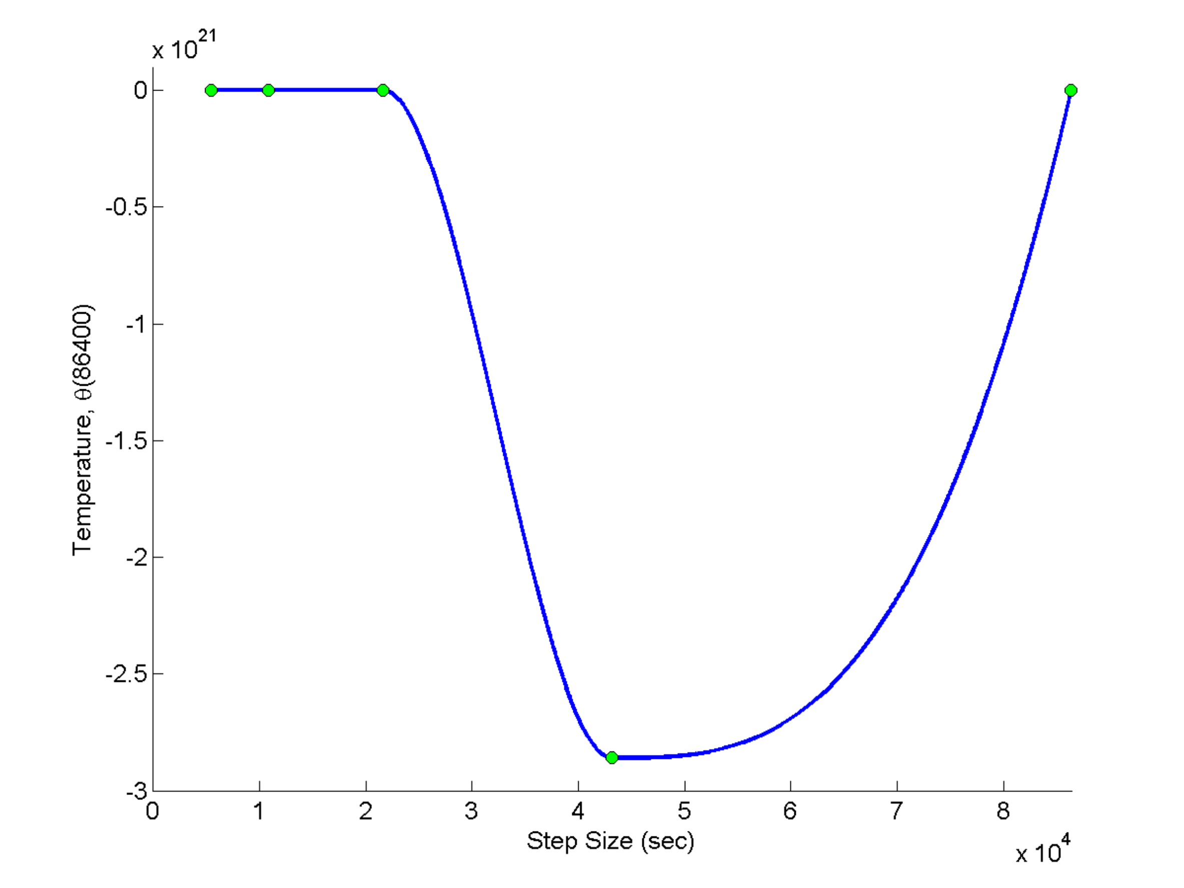


**Figure 3.** Effect of step size in Heun's method

# Comparison of Euler and Runge-Kutta 2nd Order Methods

**Table 2**. Comparison of Euler and the Runge-Kutta methods

| Step size, $h$ | $\theta(86400)$ | | | |
|---|---|---|---|---|
| | Euler | Heun | Midpoint | Ralston |
| 86400 | −153.52 | −58466 | −774.64 | −12163 |
| 43200 | −464.32 | $-2.7298\ \ 10^{21}$ | −0.33691 | −19.776 |
| 21600 | −29.541 | −24.537 | −24.069 | −24.268 |
| 10800 | −27.795 | −25.785 | −25.808 | −25.777 |
| 5400 | −26.958 | −26.027 | −26.039 | −26.032 |

$\theta(86400) = -26.099°C$ (exact)

# Comparison of Euler and Runge-Kutta 2nd Order Methods

**Table 2**. Comparison of Euler and the Runge-Kutta methods

| Step size, h | $\lvert \in_t \rvert \%$ | | | |
|---|---|---|---|---|
| | Euler | Heun | Midpoint | Ralston |
| 86400 | 448.34 | 122160 | 1027.6 | 23844 |
| 43200 | 737.97 | $7.9064 \times 10^{11}$ | 76.360 | 42.571 |
| 21600 | 14.426 | 5.7292 | 7.0508 | 6.6305 |
| 10800 | 7.0957 | 1.1993 | 1.0707 | 1.2135 |
| 5400 | 3.5755 | 0.27435 | 0.22604 | 0.25776 |

$\theta(86400) = -25.217°C$ (exact)

# Comparison of Euler and Runge-Kutta 2nd Order Methods

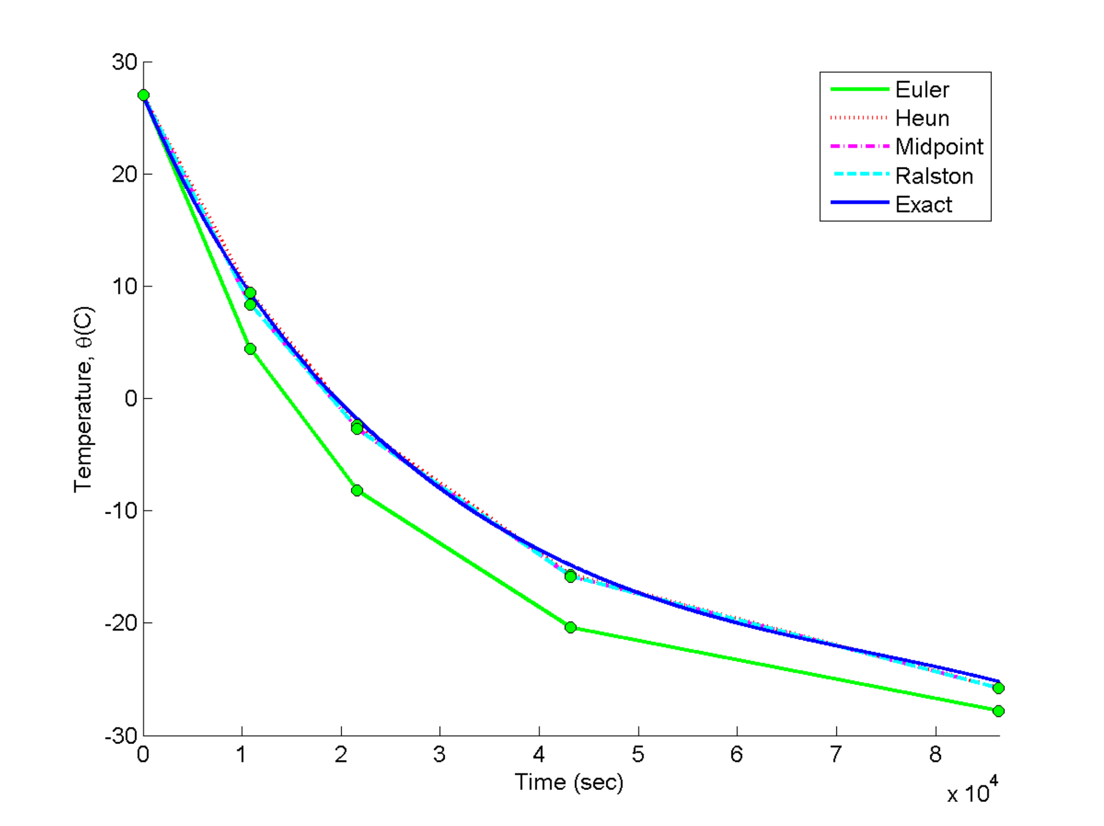


**Figure 4.** Comparison of Euler and Runge Kutta 2nd order methods with exact results.

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

# THE END

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