Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *develop Runge-Kutta 4^{th} order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4^{th} order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4^{th} order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4^{th} order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$
$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ \ y(0) = 5$$
$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2y^2}{e^{y}},\ \ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2y^2}{e^{y}}$$

The Runge-Kutta 4$^{\text{th}}$ order method is based on the following

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h \quad (1)$$

where knowing the value of $y = y_i$ at x_i, we can find the value of $y = y_{i+1}$ at x_{i+1}, and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i, y_i}(x_{i+1} - x_i)^3 + \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i, y_i}(x_{i+1} - x_i)^4 \quad (2)$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4 \quad (3)$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \quad (4)$$

$$k_1 = f(x_i, y_i) \quad (5a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \quad (5b)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \quad (5c)$$

$$k_4 = f(x_i + h, y_i + k_3 h) \quad (5d)$$

Example 3

A solid steel shaft at room temperature of $27\ ^{\mathrm{o}}\mathrm{C}$ is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at $-33\ ^{\mathrm{o}}\mathrm{C}$. The rate of change of temperature of the solid shaft θ is given by

$$\frac{d\theta}{dt} = -5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}\theta^4 + 2.33\times 10^{-5}\theta^3 + 1.35\times 10^{-3}\theta^2 + \\ +5.42\times 10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$\theta(0) = 27\ ^{\circ}\mathrm{C}$$

Using the Runge-Kutta 4th order method, find the temperature of the steel shaft after 86400 seconds. Take a step size of $h = 43200$ seconds.

Solution

$$\frac{d\theta}{dt} = -5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}\theta^4 + 2.33\times 10^{-5}\theta^3 + 1.35\times 10^{-3}\theta^2 \\ +5.42\times 10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$f(t,\theta) = -5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}\theta^4 + 2.33\times 10^{-5}\theta^3 + 1.35\times 10^{-3}\theta^2 \\ +5.42\times 10^{-2}\theta + 5.588 \end{pmatrix}(\theta + 33)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 27$

$$\begin{aligned} k_1 &= f(t_0, \theta_0) \\ &= f(0, 27) \\ &= \left(-5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}(27)^4 + 2.33\times 10^{-5}(27)^3 \\ +1.35\times 10^{-3}(27)^2 + 5.42\times 10^{-2}(27) + 5.588 \end{pmatrix}(27 + 33)\right) \\ &= -0.0020893 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) \\ &= f\left(0 + \frac{1}{2}43200, 27 + \frac{1}{2}(-0.0020893)43200\right) \\ &= f(21600, -18.129) \\ &= \left(-5.33\times 10^{-6}\begin{pmatrix} -3.69\times 10^{-6}(-18.129)^4 + 2.33\times 10^{-5}(-18.129)^3 \\ +1.35\times 10^{-3}(-18.129)^2 + 5.42\times 10^{-2}(-18.129) + 5.588 \end{pmatrix}(-18.129 + 33)\right) \\ &= -0.00035761 \end{aligned}$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right)$$

$$= f\left(0 + \frac{1}{2}43200, 27 + \frac{1}{2}(-0.00035761)43200\right)$$

$$= f(21600, 19.276)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(19.276)^4 + 2.33\times10^{-5}(19.276)^3 \\ +1.35\times10^{-3}(19.276)^2 + 5.42\times10^{-2}(19.276) + 5.588\end{pmatrix}(19.276+33)\right)$$

$$= -0.0018924$$

$$k_4 = f(t_0 + h, \theta_0 + k_3 h)$$

$$= f(0 + 43200, 27 + (-0.0018924)43200)$$

$$= f(43200, -54.751)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-54.751)^4 + 2.33\times10^{-5}(-54.751)^3 \\ +1.35\times10^{-3}(-54.751)^2 + 5.42\times10^{-2}(-54.751) + 5.588\end{pmatrix}(-54.751+33)\right)$$

$$= -0.0035147$$

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 27 + \frac{1}{6}(-0.0020893 + 2(-0.00035761) + 2(-0.0018924) + (-0.0035147))43200$$

$$= 27 + \frac{1}{6}(-0.010104)43200$$

$$= -45.749\ ^\circ\text{C}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200\ \text{s}$$

$$\theta(43200) \approx \theta_1 = -45.749\ ^\circ\text{C}$$

For $i = 1, t_1 = 43200, \theta_1 = -45.749$

$$k_1 = f(t_1, \theta_1)$$

$$= f(43200, -45.749)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-45.749)^4 + 2.33\times10^{-5}(-45.749)^3 \\ +1.35\times10^{-3}(-45.749)^2 + 5.42\times10^{-2}(-45.749) + 5.588\end{pmatrix}(-45.749+33)\right)$$

$$= -0.00084673$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(43200 + \frac{1}{2}43200, -45.749 + \frac{1}{2}(-0.00084673)43200\right)$$

$$= f(64800, -64.038)$$

$$= \left(-5.33 \times 10^{-6} \begin{pmatrix} -3.69 \times 10^{-6} (-64.038)^4 + 2.33 \times 10^{-5} (-64.038)^3 \\ +1.35 \times 10^{-3} (-64.038)^2 + 5.42 \times 10^{-2} (-64.038) + 5.588 \end{pmatrix} (-64.038 + 33) \right)$$

$$= -0.010012$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(43200 + \frac{1}{2}43200, -45.749 + \frac{1}{2}(-0.010012)43200\right)$$
$$= f(64800, -262.01)$$

$$= \left(-5.33 \times 10^{-6} \begin{pmatrix} -3.69 \times 10^{-6} (-262.01)^4 + 2.33 \times 10^{-5} (-262.01)^3 \\ +1.35 \times 10^{-3} (-262.01)^2 + 5.42 \times 10^{-2} (-262.01) + 5.588 \end{pmatrix} (-262.01 + 33) \right)$$

$$= -21.636$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h)$$
$$= f(43200 + 43200, -45.749 + (-21.636)43200)$$
$$= f(86400, -9.3474 \times 10^5)$$

$$= \left(-5.33 \times 10^{-6} \begin{pmatrix} -3.69 \times 10^{-6} (-9.4374 \times 10^5)^4 + 2.33 \times 10^{-5} (-9.4374 \times 10^5)^3 \\ +1.35 \times 10^{-3} (-9.4374 \times 10^5)^2 \\ +5.42 \times 10^{-2} (-9.4374 \times 10^5) + 5.588 \end{pmatrix} (-9.4374 \times 10^5 + 33) \right)$$

$$= -1.4035 \times 10^{19}$$

$$\theta_2 = \theta_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= -45.749 + \frac{1}{6}\begin{pmatrix} -0.00084673 + 2(-0.010012) \\ +2(-21.636) + (-1.4035 \times 10^{19}) \end{pmatrix} 43200$$
$$= -45.749 + \frac{1}{6}(-1.4035 \times 10^{19})43200$$
$$= -1.0105 \times 10^{23} \ ^\circ\text{C}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 43200 + 43200 = 86400 \text{ s}$$

$$\theta(86400) \approx \theta_2 = -1.0105 \times 10^{23}\ ^\circ\text{C}$$

The solution to this nonlinear equation at $t = 86400\,\text{s}$ is

$$\theta(86400) = -26.099\ ^\circ\text{C}$$

Figure 1 compares the exact solution with the numerical solution using Runge-Kutta 4th order method using different step sizes.

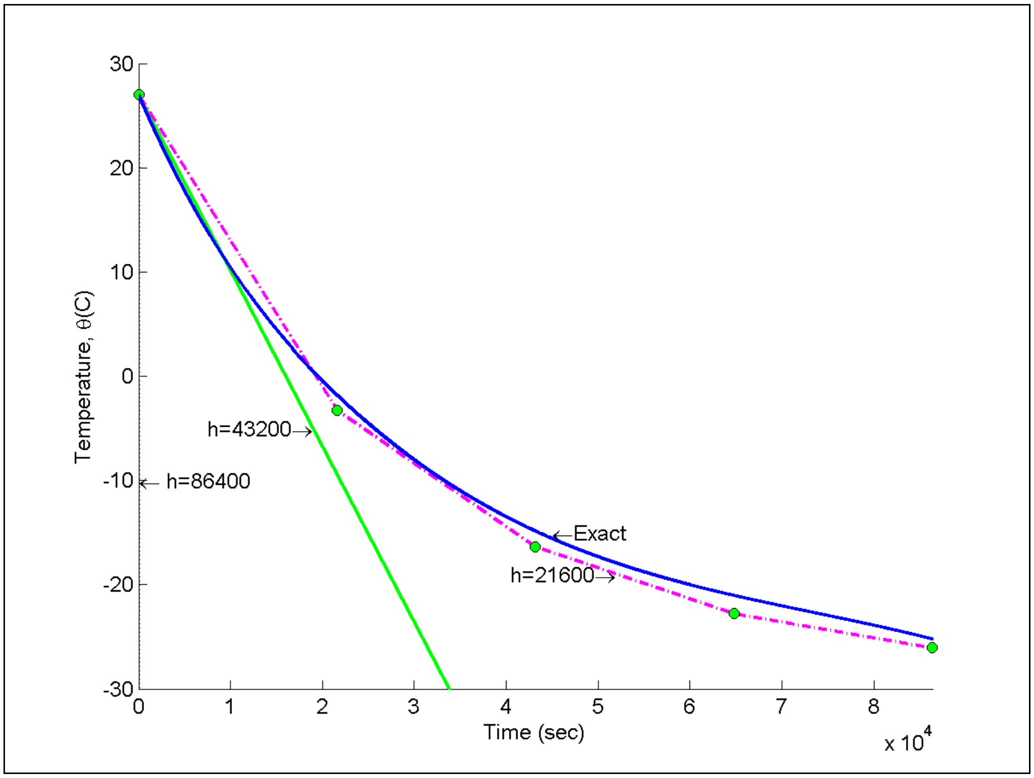

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated temperature at $t = 86400\,\text{s}$.

Table 1 Value of temperature at 86400 seconds for different step sizes.

Step size, h	$\theta(86400)$	E_t	$\lvert\in_t\rvert\,\%$
86400	-5.3468×10^{28}	-5.3468×10^{28}	2.0487×10^{29}
43200	-1.0105×10^{23}	-1.0205×10^{23}	3.8718×10^{23}
21600	−26.061	−0.038680	0.14820
10800	−26.094	−0.0050630	0.019400
5400	−26.097	−0.0015763	0.0060402

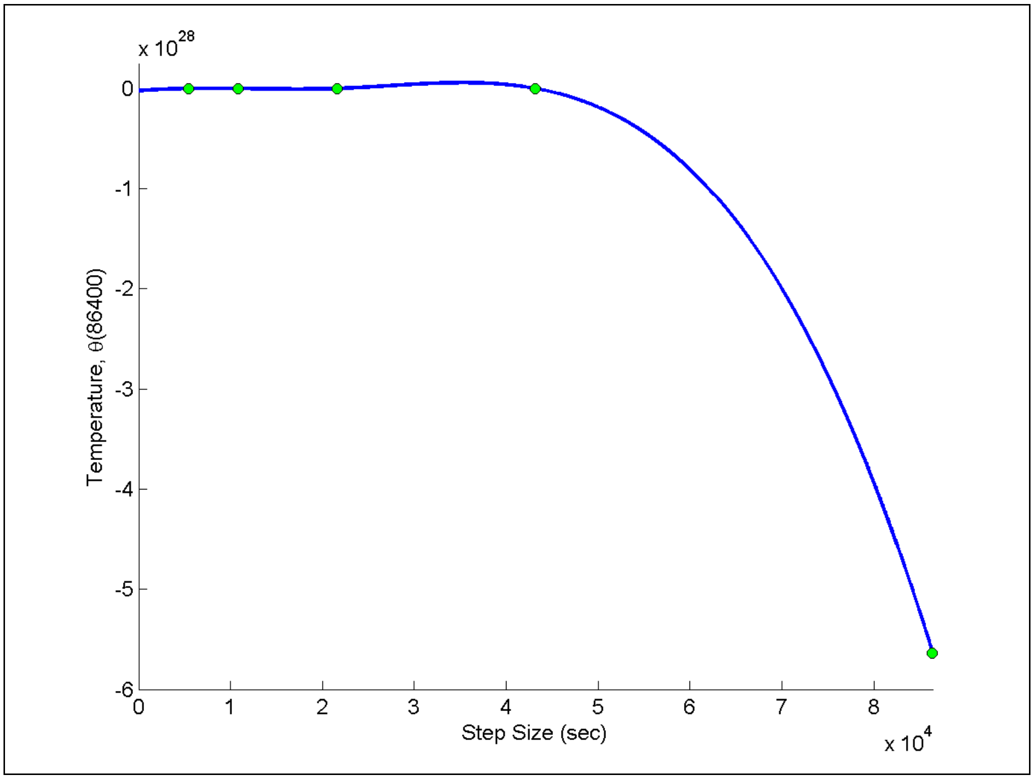

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.

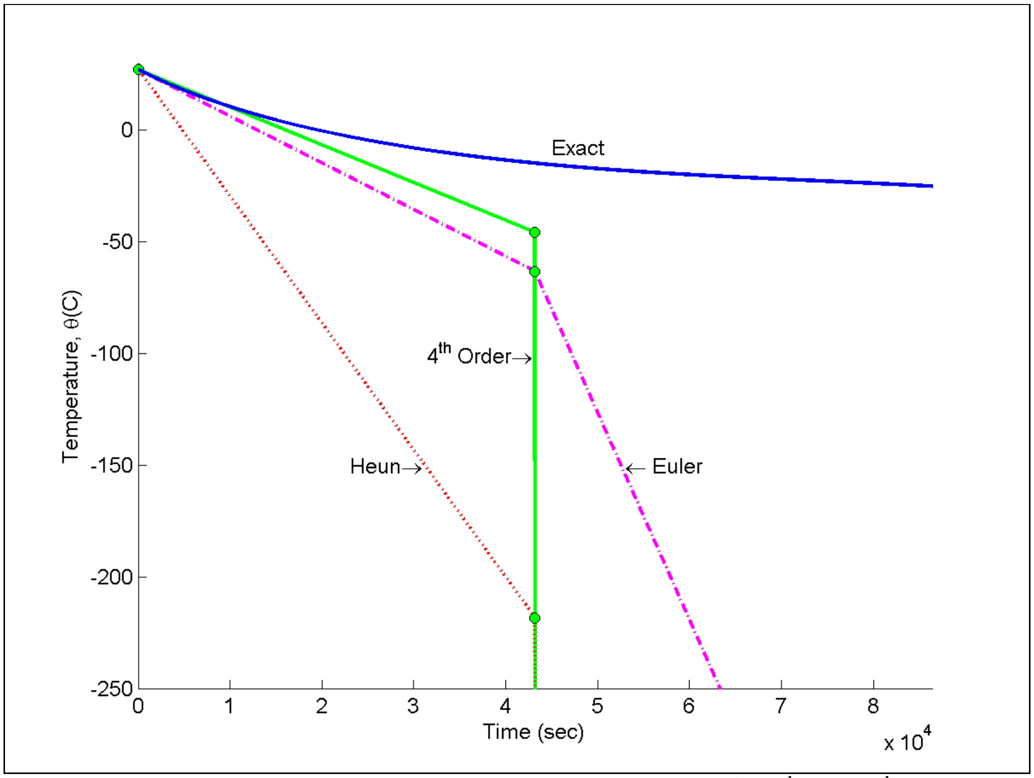

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
Major	Mechanical Engineering
Authors	Autar Kaw
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Web Site	http://numericalmethods.eng.usf.edu