Chapter 02.03
Differentiation of Discrete Functions

After reading this chapter, you should be able to:

1. *find approximate values of the first derivative of functions that are given at discrete data points, and*
2. *use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

Forward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite Δx,

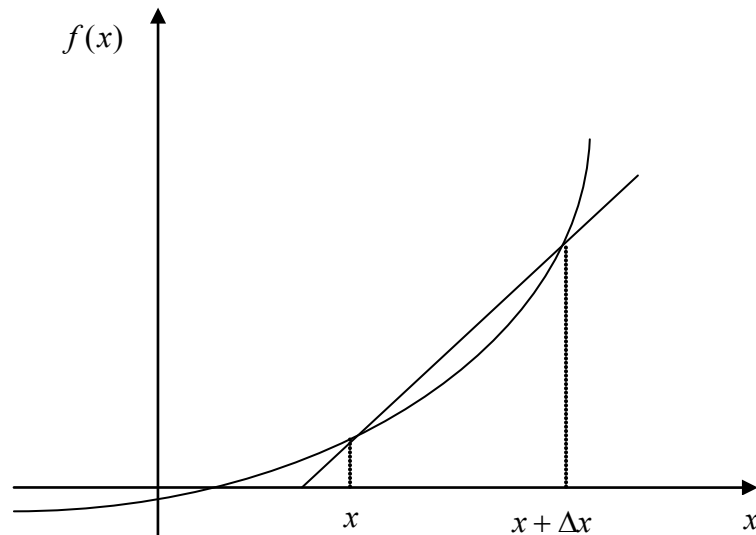$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Figure 1 Graphical representation of forward difference approximation of first derivative.

So given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the value of $f'(x)$ for $x_i \leq x \leq x_{i+1}$, $i = 0, \ldots, n-1$, is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Example 1

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 1 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,
  (a) Compute the rate at which the radius of the drop was changing at $t = 2$ seconds.
  (b) Estimate the rate at which the area of the contaminant was spreading across the pond at $t = 2$ seconds.

Table 1 Radius as a function of time.

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Radius, R (m)	0	0.236	0.667	1.225	1.886	2.635	3.464	4.365	5.333

Use the forward divided difference approximation of the first derivative to solve the above problem. Use a time step of 0.5 seconds.

Solution

(a) $R'(t_i) \approx \dfrac{R(t_{i+1}) - R(t_i)}{\Delta t}$

$$t_i = 2$$
$$t_{i+1} = 2.5$$
$$\Delta t = t_{i+1} - t_i$$
$$= 2.5 - 2$$
$$= 0.5$$
$$R'(2) \approx \frac{R(2.5) - R(2)}{0.5}$$
$$= \frac{2.635 - 1.886}{0.5}$$
$$= 1.498 \text{ m/s}$$

(b) Area $= \pi R^2$

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Area, A (m^2)	0	0.17497	1.3977	4.7144	11.175	21.813	37.697	59.857	89.350

$$A'(t_i) \approx \frac{A(t_{i+1}) - A(t_i)}{\Delta t}$$
$$t_i = 2$$
$$t_{i+1} = 2.5$$
$$\Delta t = t_{i+1} - t_i$$
$$= 2.5 - 2$$
$$= 0.5$$
$$A'(10) \approx \frac{A(2.5) - A(2)}{0.5}$$
$$= \frac{21.813 - 11.175}{0.5}$$
$$= 21.276 \text{ m}^2 / \text{s}$$

Direct Fit Polynomials

In this method, given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, one can fit a nth order polynomial given by
$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1} + a_n x^n$$
To find the first derivative,
$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1} x^{n-2} + na_n x^{n-1}$$
Similarly, other derivatives can also be found.

Example 2

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 2 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,
  (a) Compute the rate at which the radius of the drop was changing at $t = 2$ seconds.
  (b) Estimate the rate at which the area of the contaminant was spreading across the pond at $t = 2$ seconds.

Table 2 Radius as a function of time.

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Radius, R (m)	0	0.236	0.667	1.225	1.886	2.635	3.464	4.365	5.333

Use a third order polynomial interpolant for the radius and area calculations.

Solution

(a)　　For third order polynomial interpolation (also called cubic interpolation), we choose the radius given by

$$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the radius at $t = 2$, and we are using a third order polynomial, we need to choose the four points closest to $t = 2$ that also bracket $t = 2$ to evaluate it.
The four points are $t_0 = 1.0$, $t_1 = 1.5$, $t_2 = 2.0$, and $t_3 = 2.5$. (Note: Choosing $t_0 = 1.5$, $t_1 = 2.0$, $t_2 = 2.5$, and $t_3 = 3.0$ is equally valid.)

$$t_0 = 1.0, \quad R(t_0) = 0.667$$
$$t_1 = 1.5, \quad R(t_1) = 1.225$$
$$t_2 = 2.0, \quad R(t_2) = 1.886$$
$$t_3 = 2.5, \quad R(t_3) = 2.635$$

such that

$$R(1.0) = 0.667 = a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3$$
$$R(1.5) = 1.225 = a_0 + a_1(1.5) + a_2(1.5)^2 + a_3(1.5)^3$$
$$R(2.0) = 1.886 = a_0 + a_1(2.0) + a_2(2.0)^2 + a_3(2.0)^3$$
$$R(2.5) = 2.635 = a_0 + a_1(2.5) + a_2(2.5)^2 + a_3(2.5)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 2 & 4 & 8 \\ 1 & 2.5 & 6.25 & 15.625 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.667 \\ 1.225 \\ 1.886 \\ 2.635 \end{bmatrix}$$

Solving the above gives

$a_0 = -0.08$

$a_1 = 0.471$

$a_2 = 0.296$

$a_3 = -0.02$

Hence

$R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$= -0.08 + 0.471t + 0.296t^2 - 0.02t^3, \quad 1 \le t \le 2.5$
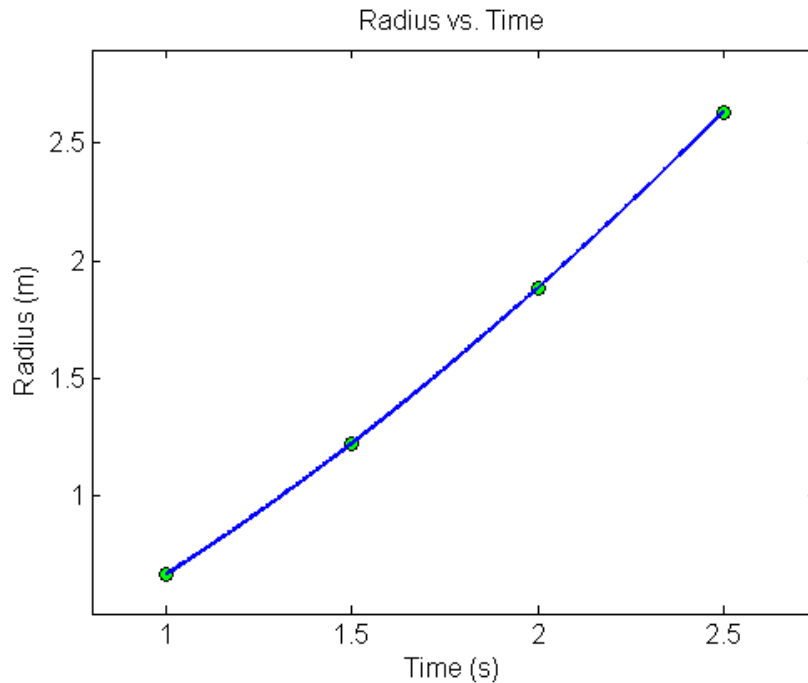

Figure 1 Graph of radius vs. time.

The derivative of the radius at $t = 2$ is given by

$$R'(2) = \frac{d}{dt} R(t)\Big|_{t=2}$$

Given that $R(t) = -0.08 + 0.471t + 0.296t^2 - 0.02t^3, \quad 1 \le t \le 2.5$,

$R'(t) = \frac{d}{dt} R(t)$

$= \frac{d}{dt}\left(-0.08 + 0.471t + 0.296t^2 - 0.02t^3\right)$

$= 0.471 + 0.592t - 0.06t^2, \quad 1 \le t \le 2.5$

$R'(2) = 0.471 + 0.592(2) - 0.06(2)^2$

$= 1.415 \text{ m/s}$

(b) Area $= \pi R^2$

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Area, A (m²)	0	0.17497	1.3977	4.7144	11.175	21.813	37.697	59.857	89.350

For third order polynomial interpolation (also called cubic interpolation), we choose the area given by

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the area at $t = 2$, and we are using a third order polynomial, we need to choose the four points closest to $t = 2$ that also bracket $t = 2$ to evaluate it.

The four points are $t_0 = 1.0$, $t_1 = 1.5$, $t_2 = 2.0$ and $t_3 = 2.5$. (Note: Choosing $t_0 = 1.5$, $t_1 = 2.0$, $t_2 = 2.5$, and $t_3 = 3.0$ is equally valid.)

$t_0 = 1.0$, $A(t_0) = 1.3977$

$t_1 = 1.5$, $A(t_1) = 4.7144$

$t_2 = 2.0$, $A(t_2) = 11.175$

$t_3 = 2.5$, $A(t_3) = 21.813$

such that

$$A(1.0) = 1.3977 = a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3$$

$$A(1.5) = 4.7144 = a_0 + a_1(1.5) + a_2(1.5)^2 + a_3(1.5)^3$$

$$A(2.0) = 11.175 = a_0 + a_1(2.0) + a_2(2.0)^2 + a_3(2.0)^3$$

$$A(2.5) = 21.813 = a_0 + a_1(2.5) + a_2(2.5)^2 + a_3(2.5)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 2 & 4 & 8 \\ 1 & 2.5 & 6.25 & 15.625 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.3977 \\ 4.7144 \\ 11.175 \\ 21.813 \end{bmatrix}$$

Solving the above gives

$a_0 = 0.057900$

$a_1 = -0.12075$

$a_2 = 0.081468$

$a_3 = 1.3790$

Hence

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

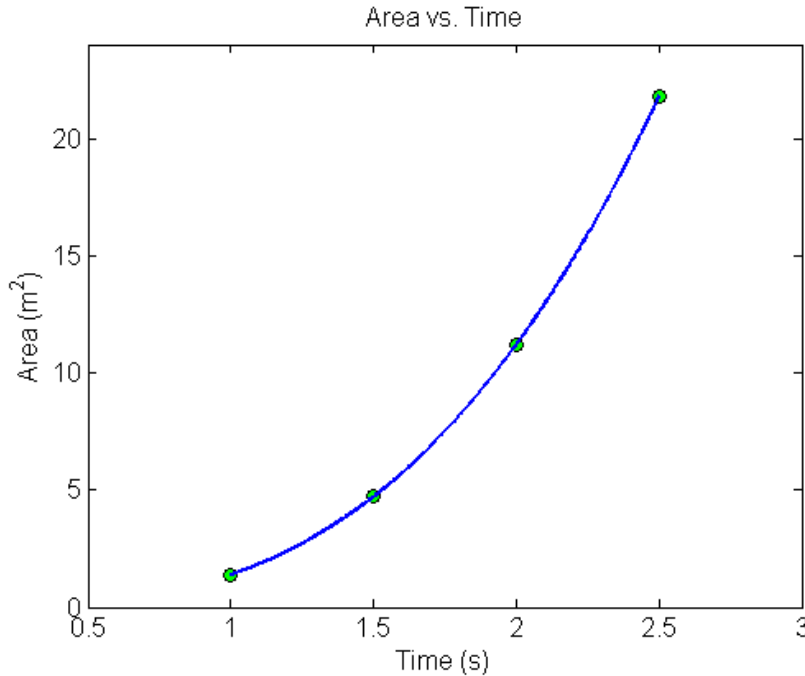$$= 0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3, \quad 1 \le t \le 2.5$$

Figure 2 Graph of area vs. time.

The derivative of the area at $t = 2$ is given by

$$A'(2) = \frac{d}{dt} A(t)\Big|_{t=2}$$

Given that $A(t) = 0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3$, $\quad 1 \le t \le 2.5$,

$$\begin{aligned}
A'(t) \ &= \frac{d}{dt} A(t) \\
&= \frac{d}{dt}\left(0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3\right) \\
&= -0.12075 + 0.16294t + 4.1371t^2, \quad 1 \le t \le 2.5
\end{aligned}$$

$$\begin{aligned}
A'(2) &= -0.12075 + 0.16294(2) + 4.1371(2)^2 \\
&= 16.754 \text{ m}^2/\text{s}
\end{aligned}$$

Lagrange Polynomial

In this method, given $(x_0, y_0), \ldots, (x_n, y_n)$, one can fit a nth order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the nth order polynomial that approximates the function $y = f(x)$ and

$$L_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j=i$ omitted.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1),$ and (x_2, y_2) is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Example 3

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 3 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,
   (a) Compute the rate at which the radius of the drop was changing at $t=2$ seconds.
   (b) Estimate the rate at which the area of the contaminant was spreading across the pond at $t=2$ seconds.

Table 3 Radius as a function of time.

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Radius, R (m)	0	0.236	0.667	1.225	1.886	2.635	3.464	4.365	5.333

Use second order Lagrangian polynomial interpolation to solve the problem.

Solution

(a) For second order Lagrangian polynomial interpolation, we choose the radius given by

$$R(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)R(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)R(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)R(t_2)$$

Since we want to find the radius at $t = 2$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t = 2$ that also bracket $t = 2$ to evaluate it.

The three points are $t_0 = 1.5$, $t_1 = 2.0$, and $t_2 = 2.5$.

Differentiating the above equation gives

$$R'(t) = \frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}R(t_0) + \frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}R(t_1) + \frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}R(t_2)$$

Hence

$$R'(2) = \frac{2(2)-(2.0+2.5)}{(1.5-2.0)(1.5-2.5)}(1.225) + \frac{2(2)-(1.5+2.5)}{(2.0-1.5)(2.0-2.5)}(1.886)$$
$$+ \frac{2(2)-(1.5+2.0)}{(2.5-1.5)(2.5-2.0)}(2.635)$$
$$= 1.41 \, \text{m/s}$$

(b) Area $= \pi R^2$

Time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Area, A (m^2)	0	0.17497	1.3977	4.7144	11.175	21.813	37.697	59.857	89.350

For second order Lagrangian polynomial interpolation, we choose the area given by

$$A(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)A(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)A(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)A(t_2)$$

Since we want to find the area at $t = 2$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t = 2$ that also bracket $t = 2$ to evaluate it. The three points are $t_0 = 1.5$, $t_1 = 2.0$, and $t_2 = 2.5$.

Differentiating the above equation gives

$$A'(t) = \frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}A(t_0) + \frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}A(t_1) + \frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}A(t_2)$$

Hence

$$A'(2) = \frac{2(2)-(2.0+2.5)}{(1.5-2.0)(1.5-2.5)}(4.7144) + \frac{2(2)-(1.5+2.5)}{(2.0-1.5)(2.0-2.5)}(11.175)$$
$$+ \frac{2(2)-(1.5+2.0)}{(2.5-1.5)(2.5-2.0)}(21.813)$$
$$= 17.099 \, \text{m}^2/\text{s}$$

DIFFERENTIATION	
Topic	Differentiation of Discrete Functions
Summary	These are textbook notes differentiation of discrete functions
Major	Chemical Engineering
Authors	Autar Kaw, Luke Snyder
Date	November 8, 2012
Web Site	http://numericalmethods.eng.usf.edu