Lagrangian Interpolation

Chemical Engineering Majors

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Lagrange Method of Interpolation

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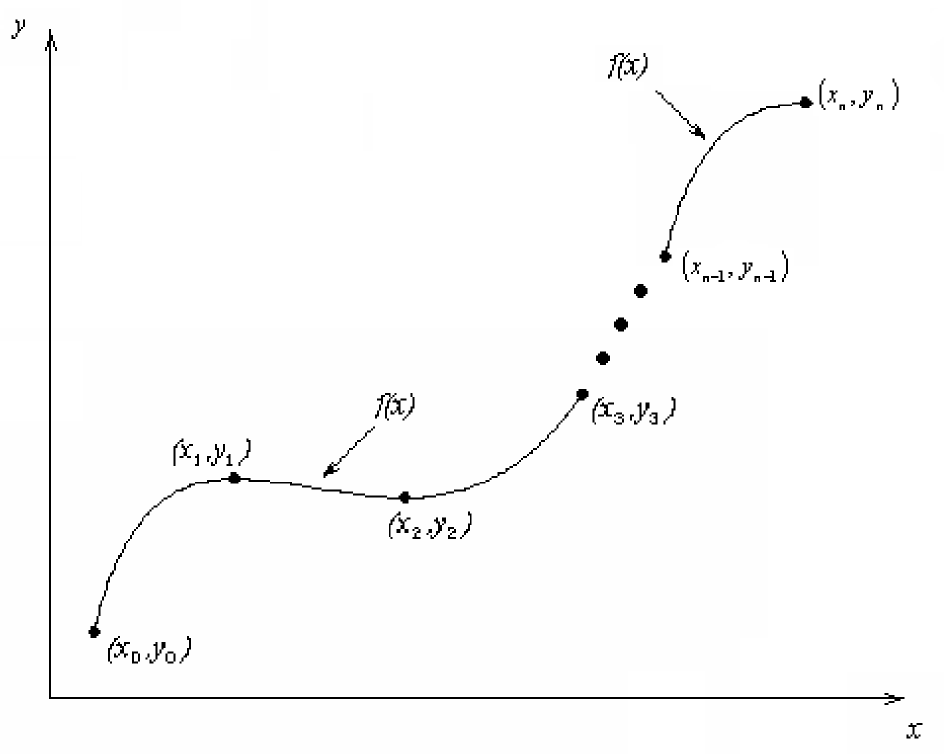
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

To find how much heat is required to bring a kettle of water to boiling point, you are asked to calculate the specific heat of water at 61^0C. Use a first, second and third order Lagrange polynomial to determine the value of the specific heat at T = 61˚C.

Table 1 Specific heat of water as a function of temperature.

Temperature, $T\ (°C)$	Specific heat, $C_p \left(\frac{J}{kg-°C}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

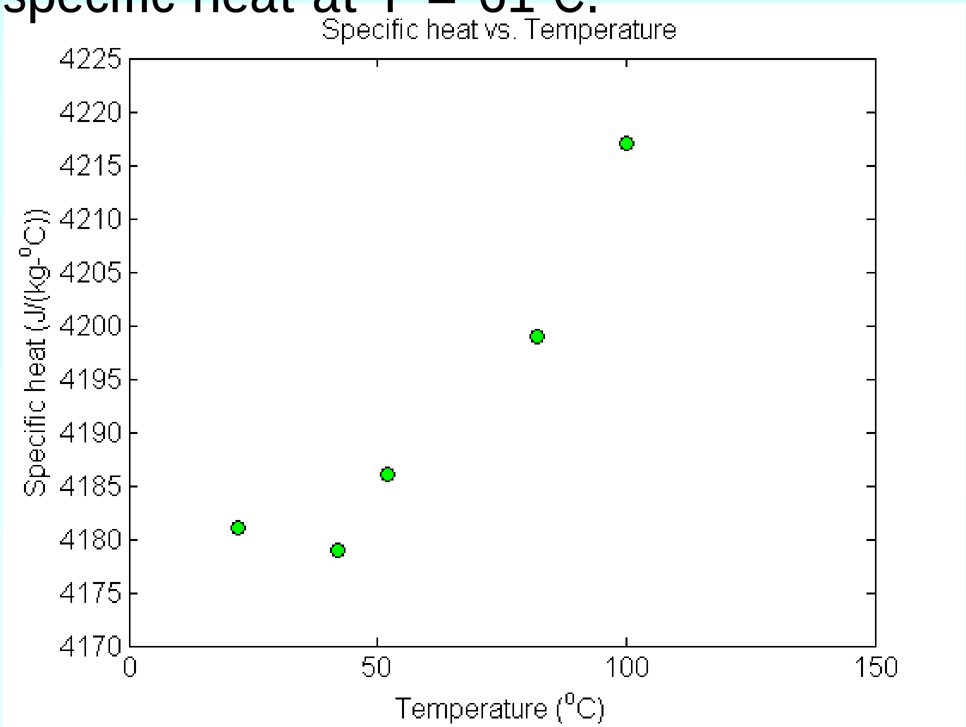


Figure 2 Specific heat of water vs. temperature.

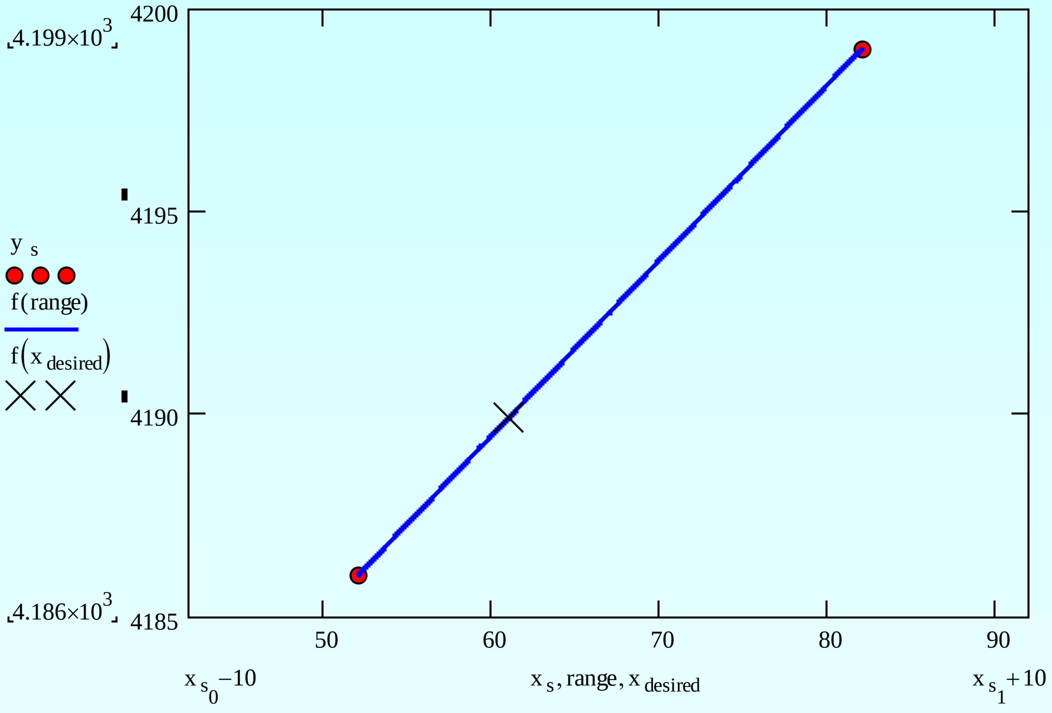
Linear Interpolation

$$C_p(T) = \sum_{i=0}^{1} L_i(T)\,C_p(T_i)$$

$$= L_0(T)\,C_p(T_0) + L_1(T)\,C_p(T_1)$$

$$T_0 = 52,\; C_p(T_0) = 4186$$

$$T_1 = 82,\; C_p(T_1) = 4199$$



Linear Interpolation (contd)

$$L_0(T)=\prod_{\substack{j=0\\ j\neq 0}}^{1}\frac{T-T_j}{T_0-T_j}=\frac{T-T_1}{T_0-T_1}$$

$$L_1(T)=\prod_{\substack{j=0\\ j\neq 1}}^{1}\frac{T-T_j}{T_1-T_j}=\frac{T-T_0}{T_1-T_0}$$

$$Cp(T)=\frac{T-T_1}{T_0-T_1}Cp(T_0)+\frac{T-T_0}{T_1-T_0}Cp(T_1)=\frac{T-82}{52-82}(4186)+\frac{T-52}{82-52}(4199),52\leq T\leq 82$$

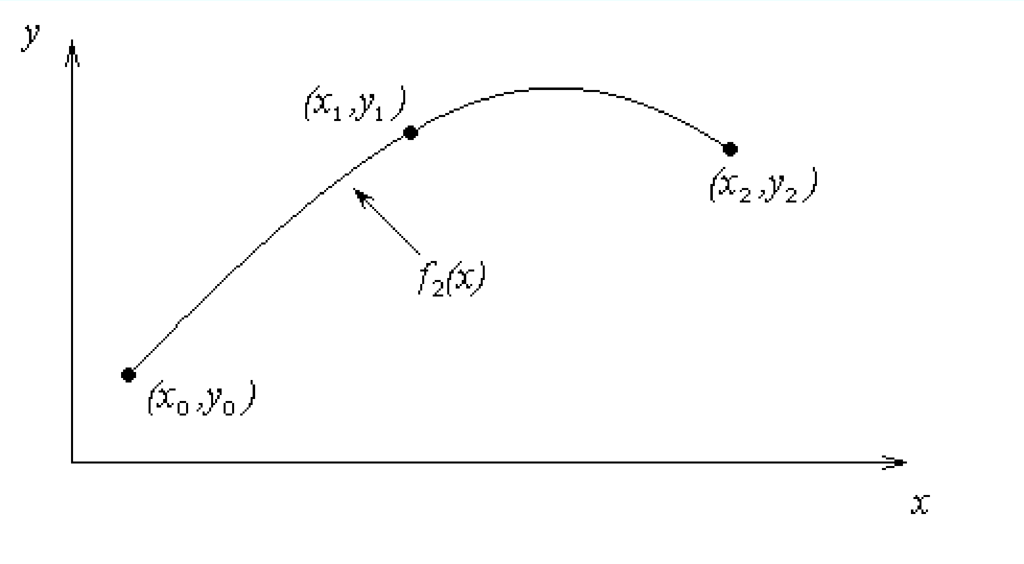
$$Cp(61)=\frac{(61)-82}{52-82}(4186)+\frac{(61)-52}{82-52}(4199)=4189.9\frac{J}{kg-{}^\circ C}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the specific heat given by

$$C_p(T) = \sum_{i=0}^{2} L_i(T) C_p(T_i)$$

$$= L_0(T) C_p(T_0) + L_1(T) C_p(T_1) + L_2(T) C_p(T_2)$$



Quadratic Interpolation (contd)

$$T_0 = 42, \ \ C_p(T_0) = 4179$$

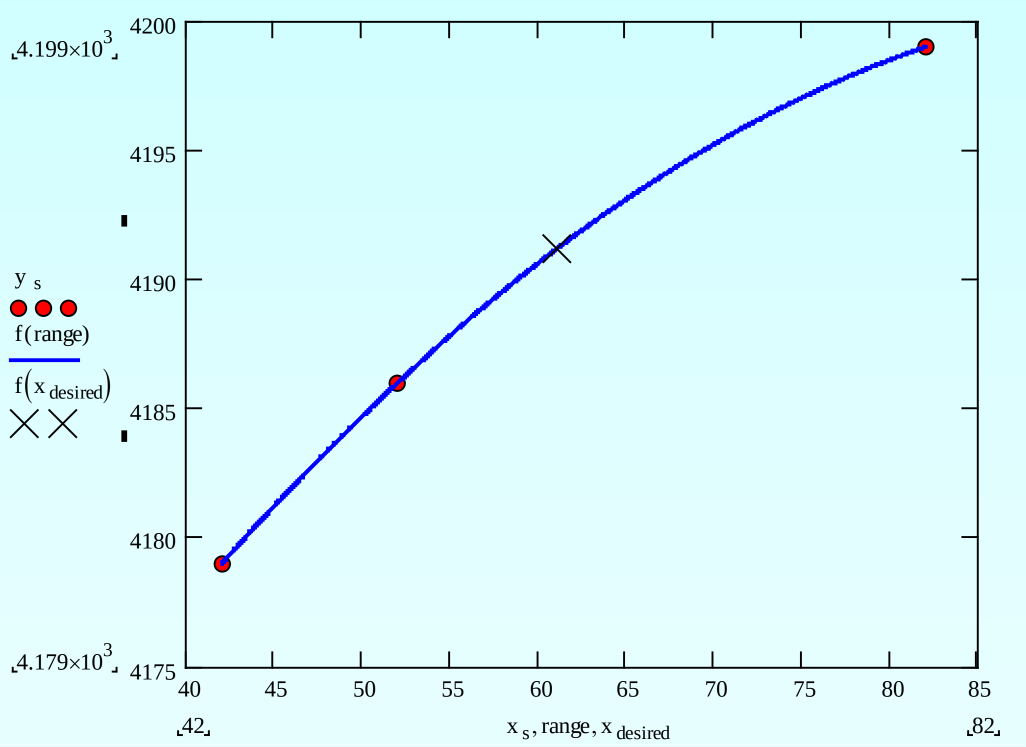
$$T_1 = 52, \ \ C_p(T_1) = 4186$$

$$T_2 = 82, \ \ C_p(T_2) = 4199$$

$$L_0(T) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{T - T_j}{T_0 - T_j} = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{T - T_j}{T_1 - T_j} = \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{T - T_j}{T_2 - T_j} = \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)$$



Quadratic Interpolation (contd)

$$C_p(T)=\left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)C_p(T_0)+\left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)C_p(T_1)+\left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)C_p(T_2)$$

$$T_1 \le T \le T_2$$

$$C_p(61)=\frac{(61-52)(61-82)}{(42-52)(42-82)}(4179)+\frac{(61-42)(61-82)}{(52-42)(52-82)}(4186)+\frac{(61-42)(61-52)}{(82-42)(82-52)}(4199)$$

$$=(-0.4725)(4179)+(1.33)(4186)+(0.1425)(4199)$$

$$=4191.2\frac{J}{kg-^\circ C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{4191.2-4189.9}{4191.2}\right|\times 100$$

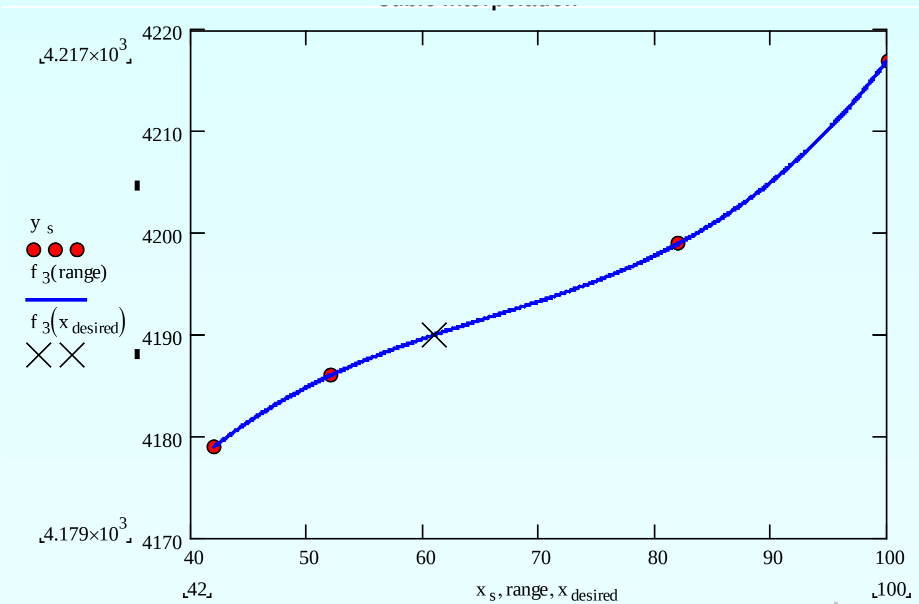
$$=0.030063\%$$

Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the specific heat given by

$$C_p(T) = \sum_{i=0}^{3} L_i(T) C_p(T_i)$$

$$= L_0(T)C_p(T_0) + L_1(T)C_p(T_1) + L_2(T)C_p(T_2) + L_3(T)C_p(T_3)$$



Cubic Interpolation (contd)

$$T_o = 42, \; C_p(T_o) = 4179$$

$$T_1 = 52, \; C_p(T_1) = 4186$$

$$T_2 = 82, \; C_p(T_2) = 4199$$

$$T_3 = 100, \; C_p(T_3) = 4217$$

$$L_0(T) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{T - T_j}{T_0 - T_j} = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\left(\frac{T - T_3}{T_0 - T_3}\right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j \neq 1}}^{3} \frac{T - T_j}{T_1 - T_j} = \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\left(\frac{T - T_3}{T_1 - T_3}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j \neq 2}}^{3} \frac{T - T_j}{T_2 - T_j} = \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\left(\frac{T - T_3}{T_2 - T_3}\right)$$

$$L_3(T) = \prod_{\substack{j=0 \\ j \neq 3}}^{3} \frac{T - T_j}{T_3 - T_j} = \left(\frac{T - T_0}{T_3 - T_0}\right)\left(\frac{T - T_1}{T_3 - T_1}\right)\left(\frac{T - T_2}{T_3 - T_2}\right)$$

Cubic Interpolation (contd)

$$C_p(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\left(\frac{T-T_3}{T_0-T_3}\right)C_p(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\left(\frac{T-T_3}{T_1-T_3}\right)C_p(T_1)$$

$$+\left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\left(\frac{T-T_3}{T_2-T_3}\right)C_p(T_2) + \left(\frac{T-T_0}{T_3-T_0}\right)\left(\frac{T-T_1}{T_3-T_1}\right)\left(\frac{T-T_2}{T_3-T_2}\right)C_p(T_3)$$

$$C_p(61) = \frac{(61-52)(61-82)(61-100)}{(42-52)(42-82)(42-100)}(4179) + \frac{(61-42)(61-82)(61-100)}{(52-42)(52-82)(52-100)}(4186)$$

$$+\frac{(61-42)(61-52)(61-100)}{(82-42)(82-52)(82-100)}(4199) + \frac{(61-42)(61-52)(61-82)}{(100-42)(100-52)(100-82)}(4217)$$

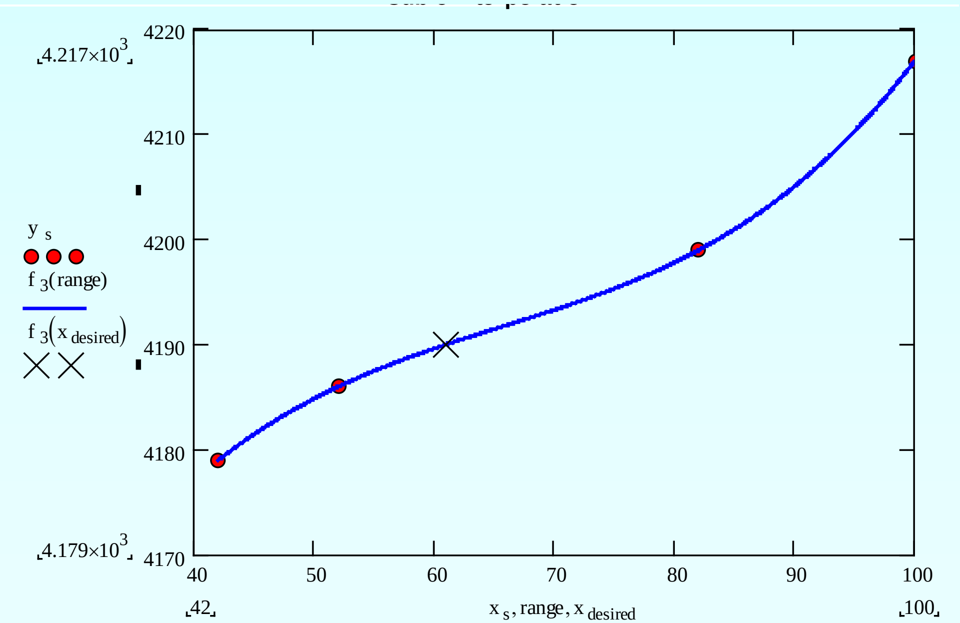
$$= (-0.31772)(4179) + (1.0806)(4186) + (0.30875)(4199) + (-0.071659)(4217)$$

$$= 4190.0\,\frac{J}{kg-^{\circ}C}$$

Cubic Interpolation (contd)

The absolute relative approximate error obtained between the results from the second and third order polynomial is

$$\left|\in_a\right| = \left|\frac{4190.0 - 4191.2}{4190.0}\right| \times 100 = 0.027295\%$$



Comparison Table

Order of Polynomial	1	2	3
$C_p(T)\,\dfrac{J}{kg-^{\circ}C}$	4189.9	4191.2	4190.0
Absolute Relative Approximate Error	----------	0.030063%	0.027295%

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/lagrange_method.html

THE END

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