

Chapter 05.04

Lagrangian Interpolation

After reading this chapter, you should be able to:

1. derive Lagrangian method of interpolation,
2. solve problems using Lagrangian method of interpolation, and
3. use Lagrangian interpolants to find derivatives and integrals of discrete functions.

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0) , (x_1, y_1) , \dots , (x_{n-1}, y_{n-1}) , (x_n, y_n) . So, how then does one find the value of y at any other value of x ? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x . This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

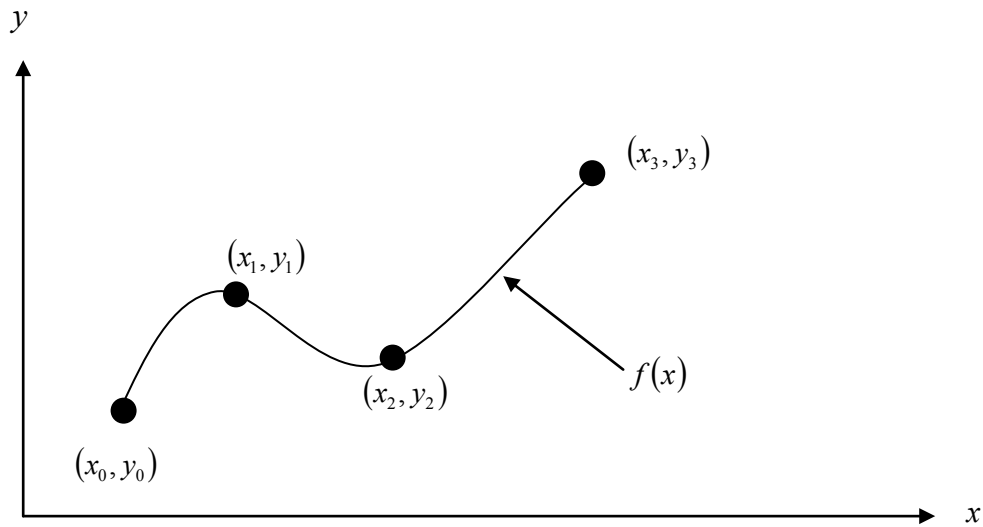


Figure 1 Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n + 1$ data points as $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

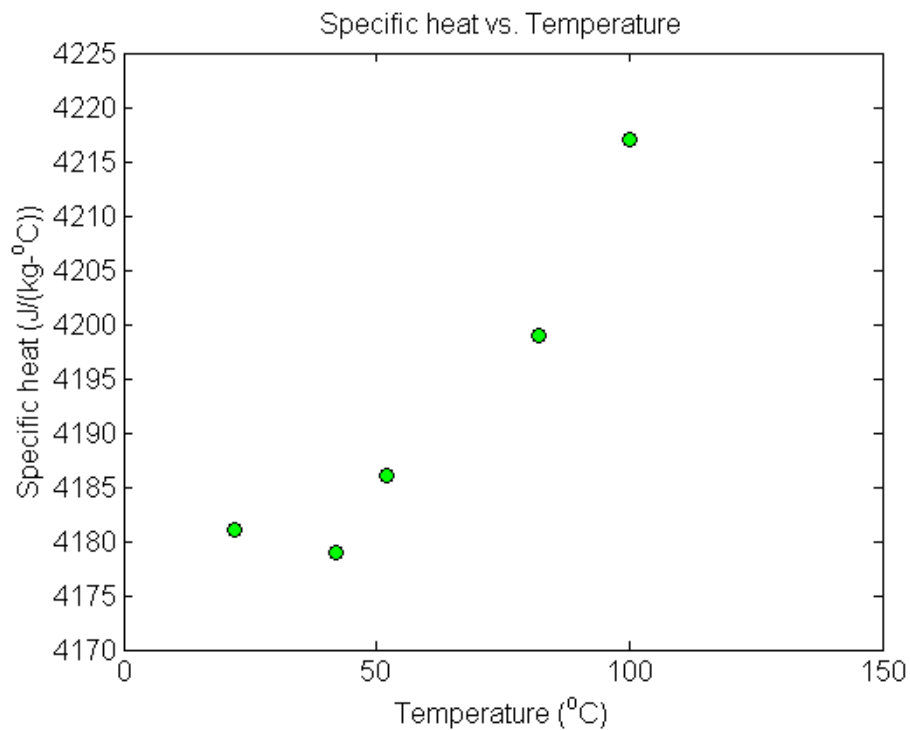
$L_i(x)$ is a weighting function that includes a product of $n - 1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C . The specific heat of water is given as a function of time in Table 1.

Table 1 Specific heat of water as a function of temperature.

Temperature, T ($^{\circ}\text{C}$)	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} \cdot ^{\circ}\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

**Figure 2** Specific heat of water vs. temperature.

Determine the value of the specific heat at $T = 61^{\circ}\text{C}$ using a first order Lagrange polynomial.

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the specific heat is given by

$$\begin{aligned}
 C_p(T) &= \sum_{i=0}^1 L_i(T)C_p(T_i) \\
 &= L_0(T)C_p(T_0) + L_1(T)C_p(T_1)
 \end{aligned}$$

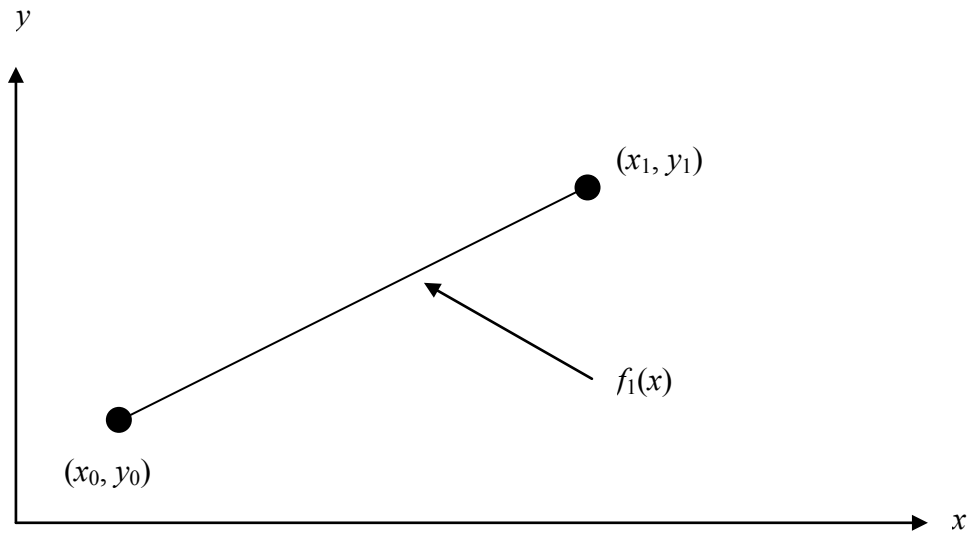


Figure 3 Linear interpolation.

Since we want the velocity at $T = 61^\circ\text{C}$, we need to choose the two data points that are closest to $T = 61^\circ\text{C}$ that also bracket $T = 61^\circ\text{C}$ to evaluate it. The two points are $T_0 = 52$ and $T_1 = 82$.

Then

$$T_0 = 52, C_p(T_0) = 4186$$

$$T_1 = 82, C_p(T_1) = 4199$$

gives

$$\begin{aligned} L_0(T) &= \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{T - T_j}{T_0 - T_j} \\ &= \frac{T - T_1}{T_0 - T_1} \end{aligned}$$

$$\begin{aligned} L_1(T) &= \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{T - T_j}{T_1 - T_j} \\ &= \frac{T - T_0}{T_1 - T_0} \end{aligned}$$

Hence

$$\begin{aligned} C_p(T) &= \frac{T - T_1}{T_0 - T_1} C_p(T_0) + \frac{T - T_0}{T_1 - T_0} C_p(T_1) \\ &= \frac{T - 82}{52 - 82} (4186) + \frac{T - 52}{82 - 52} (4199), \quad 52 \leq T \leq 82 \end{aligned}$$

$$\begin{aligned}
 C_p(61) &= \frac{61-82}{52-82}(4186) + \frac{61-52}{82-52}(4199) \\
 &= 0.7(4186) + 0.3(4199) \\
 &= 4189.9 \frac{\text{J}}{\text{kg} - ^\circ\text{C}}
 \end{aligned}$$

You can see that $L_0(T) = 0.7$ and $L_1(T) = 0.3$ are like weightages given to the specific heats at $T = 52$ and $T = 82$ to calculate the specific heat at $T = 61$.

Example 2

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C . The specific heat of water is given as a function of time in Table 2.

Table 2 Specific heat of water as a function of temperature.

Temperature, T ($^\circ\text{C}$)	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61^\circ\text{C}$ using a second order Lagrange polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the specific heat given by

$$\begin{aligned}
 C_p(T) &= \sum_{i=0}^2 L_i(T)C_p(T_i) \\
 &= L_0(T)C_p(T_0) + L_1(T)C_p(T_1) + L_2(T)C_p(T_2)
 \end{aligned}$$

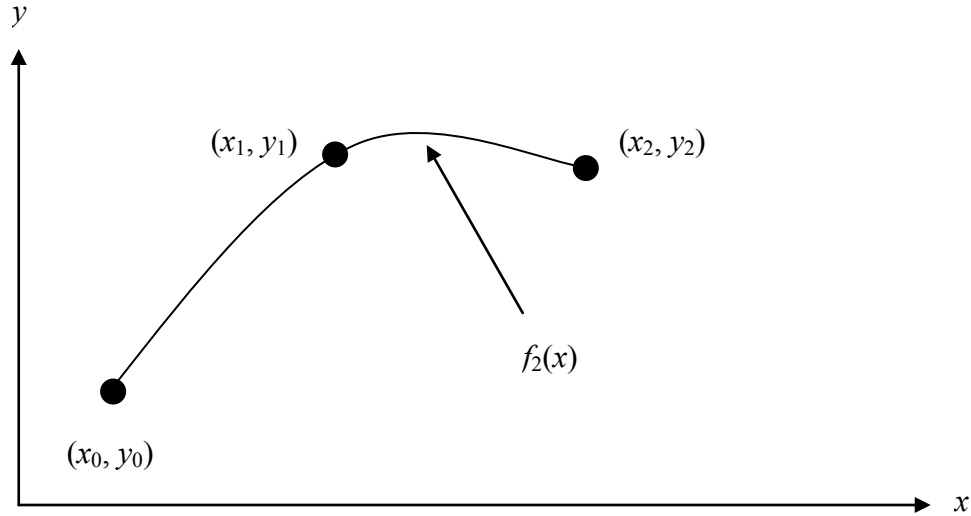


Figure 4 Quadratic interpolation.

Since we want to find the specific heat at $T = 61^\circ\text{C}$, we need to choose the three data points that are closest to $T = 61^\circ\text{C}$ that also bracket $T = 61^\circ\text{C}$ to evaluate it. The three points are $T_0 = 42$, $T_1 = 52$ and $T_2 = 82$.

Then

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

gives

$$\begin{aligned} L_0(T) &= \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{T - T_j}{T_0 - T_j} \\ &= \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \end{aligned}$$

$$\begin{aligned} L_1(T) &= \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{T - T_j}{T_1 - T_j} \\ &= \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \end{aligned}$$

$$\begin{aligned} L_2(T) &= \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{T - T_j}{T_2 - T_j} \\ &= \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \end{aligned}$$

Hence

$$C_p(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)C_p(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)C_p(T_1) + \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)C_p(T_2),$$

$$T_1 \leq T \leq T_2$$

$$C_p(61) = \frac{(61-52)(61-82)}{(42-52)(42-82)}(4179) + \frac{(61-42)(61-82)}{(52-42)(52-82)}(4186) + \frac{(61-42)(61-52)}{(82-42)(82-52)}(4199)$$

$$= (-0.4725)(4179) + (1.33)(4186) + (0.1425)(4199)$$

$$= 4191.2 \frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$|\epsilon_a| = \left| \frac{4191.2 - 4189.9}{4191.2} \right| \times 100$$

$$= 0.030063\%$$

Example 3

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C . The specific heat of water is given as a function of time in Table 3.

Table 3 Specific heat of water as a function of temperature.

Temperature, T ($^\circ\text{C}$)	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61^\circ\text{C}$ using a third order Lagrange polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

Solution

For third order Lagrange polynomial interpolation (also called cubic interpolation), we choose the specific heat given by

$$C_p(T) = \sum_{i=0}^3 L_i(T)C_p(T_i)$$

$$= L_0(T)C_p(T_0) + L_1(T)C_p(T_1) + L_2(T)C_p(T_2) + L_3(T)C_p(T_3)$$

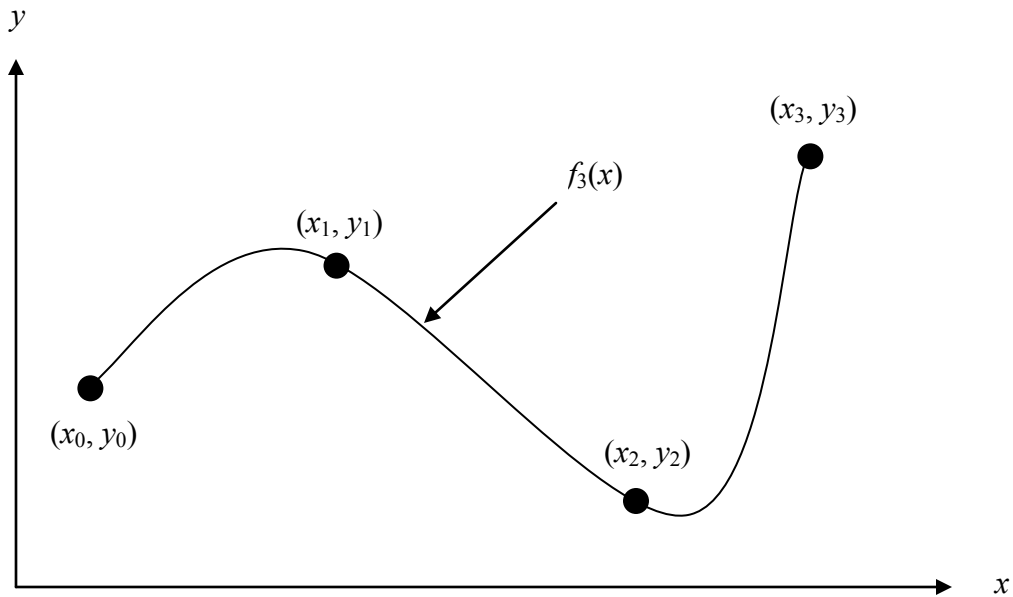


Figure 5 Cubic interpolation.

Since we wish to find the velocity at $T = 61^\circ\text{C}$, we need to choose four data points that are closest to $T = 61^\circ\text{C}$ and bracket $T = 61^\circ\text{C}$ to evaluate it. The four data points are $T_0 = 42$, $T_1 = 52$, $T_2 = 82$ and $T_3 = 100$. (Choosing the four points as $T_0 = 22$, $T_1 = 42$, $T_2 = 52$ and $T_3 = 82$ is equally valid.)

Then

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

$$T_3 = 100, \quad C_p(T_3) = 4217$$

gives

$$L_0(T) = \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{T - T_j}{T_0 - T_j}$$

$$= \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \left(\frac{T - T_3}{T_0 - T_3} \right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{T - T_j}{T_1 - T_j}$$

$$= \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \left(\frac{T - T_3}{T_1 - T_3} \right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{T - T_j}{T_2 - T_j}$$

$$= \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \left(\frac{T - T_3}{T_2 - T_3} \right)$$

$$L_3(T) = \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{T - T_j}{T_3 - T_j}$$

$$= \left(\frac{T - T_0}{T_3 - T_0} \right) \left(\frac{T - T_1}{T_3 - T_1} \right) \left(\frac{T - T_2}{T_3 - T_2} \right)$$

Hence

$$C_p(T) = \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \left(\frac{T - T_3}{T_0 - T_3} \right) C_p(T_0) + \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \left(\frac{T - T_3}{T_1 - T_3} \right) C_p(T_1)$$

$$+ \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \left(\frac{T - T_3}{T_2 - T_3} \right) C_p(T_2) + \left(\frac{T - T_0}{T_3 - T_0} \right) \left(\frac{T - T_1}{T_3 - T_1} \right) \left(\frac{T - T_2}{T_3 - T_2} \right) C_p(T_3)$$

$T_0 \leq T \leq T_3$

$$v(16) = \frac{(61 - 52)(61 - 82)(61 - 100)}{(42 - 52)(42 - 82)(42 - 100)} (4179) + \frac{(61 - 42)(61 - 82)(61 - 100)}{(52 - 42)(52 - 82)(52 - 100)} (4186)$$

$$+ \frac{(61 - 42)(61 - 52)(61 - 100)}{(82 - 42)(82 - 52)(82 - 100)} (4199) + \frac{(61 - 42)(61 - 52)(61 - 82)}{(100 - 42)(100 - 52)(100 - 82)} (4217)$$

$$= (-0.31772)(4179) + (1.0806)(4186) + (0.30875)(4199) + (-0.071659)(4217)$$

$$= 4190.0 \frac{\text{J}}{\text{kg} - ^\circ\text{C}}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the second and third order polynomial is

$$|\epsilon_a| = \left| \frac{4190.0 - 4191.2}{4190.0} \right| \times 100$$

$$= 0.027295\%$$

INTERPOLATION

Topic	Lagrange Interpolation
Summary	Textbook notes on the Lagrangian method of interpolation
Major	Chemical Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	November 8, 2012
Web Site	http://numericalmethods.eng.usf.edu
