

## Chapter 07.03

### Simpson's 1/3 Rule of Integration

*After reading this chapter, you should be able to*

1. *derive the formula for Simpson's 1/3 rule of integration,*
2. *use Simpson's 1/3 rule to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 1/3 rule of integration,*
4. *use multiple-segment Simpson's 1/3 rule of integration to solve integrals, and*
5. *derive the true error formula for multiple-segment Simpson's 1/3 rule.*

#### **What is integration?**

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x) dx$$

where

$f(x)$  is called the integrand,

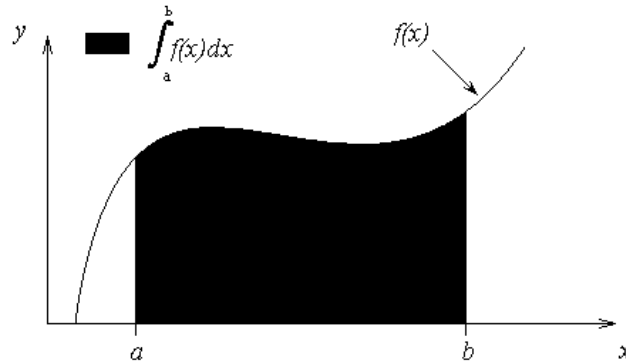
$a$  = lower limit of integration

$b$  = upper limit of integration

#### **Simpson's 1/3 Rule**

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an

extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.



**Figure 1** Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x) dx \approx \int_a^b f_2(x) dx$$

where  $f_2(x)$  is a second order polynomial given by

$$f_2(x) = a_0 + a_1x + a_2x^2$$

Choose

$$(a, f(a)), \left( \frac{a+b}{2}, f\left(\frac{a+b}{2}\right) \right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate  $a_0$ ,  $a_1$  and  $a_2$ .

$$f(a) = f_2(a) = a_0 + a_1a + a_2a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1b + a_2b^2$$

Solving the above three equations for unknowns,  $a_0$ ,  $a_1$  and  $a_2$  give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$\begin{aligned} I &\approx \int_a^b f_2(x) dx \\ &= \int_a^b (a_0 + a_1x + a_2x^2) dx \\ &= \left[ a_0x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right]_a^b \\ &= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3} \end{aligned}$$

Substituting values of  $a_0$ ,  $a_1$  and  $a_2$  give

$$\int_a^b f_2(x) dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since for Simpson 1/3 rule, the interval  $[a, b]$  is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x) dx \approx \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

#### Method 2:

Simpson's 1/3 rule can also be derived by approximating  $f(x)$  by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$\begin{aligned} b_0 &= f(a) \\ b_1 &= \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} \end{aligned}$$

$$b_2 = \frac{\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b - a}$$

Integrating Newton's divided difference polynomial gives us

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b f_2(x) dx \\ &= \int_a^b \left[ b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right) \right] dx \\ &= \left[ b_0 x + b_1 \left( \frac{x^2}{2} - ax \right) + b_2 \left( \frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2} \right) \right]_a^b \\ &= b_0(b-a) + b_1 \left( \frac{b^2 - a^2}{2} - a(b-a) \right) \\ &\quad + b_2 \left( \frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2} \right) \end{aligned}$$

Substituting values of  $b_0$ ,  $b_1$ , and  $b_2$  into this equation yields the same result as before

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \end{aligned}$$

### Method 3:

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \frac{a+b}{2}\right)(x-b)}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \frac{a+b}{2}\right)}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\begin{aligned}
 \int_a^b f_2(x) dx &= \left[ \frac{x^3 - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{x^3 - \frac{(a+b)x^2}{2} + abx}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) \right]_a^b \\
 &+ \left[ \frac{x^3 - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b) \right]_a^b \\
 &= \frac{b^3 - a^3}{3} - \frac{(a+3b)(b^2 - a^2)}{4} + \frac{b(a+b)(b-a)}{2} f(a) \\
 &+ \frac{b^3 - a^3}{3} - \frac{(a+b)(b^2 - a^2)}{2} + ab(b-a) f\left(\frac{a+b}{2}\right) \\
 &+ \frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2} f(b)
 \end{aligned}$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\begin{aligned}
 \int_a^b f(x) dx &\approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\
 &= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].
 \end{aligned}$$

#### Method 4:

Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x) dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals  $\int_a^b 1 dx$ ,  $\int_a^b x dx$ , and  $\int_a^b x^2 dx$ . This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1 dx = b - a = c_1 + c_2 + c_3$$

$$\int_a^b x dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left( \frac{a+b}{2} \right)^2 + c_3 b^2$$

Solving the above three equations for  $c_0$ ,  $c_1$  and  $c_2$  give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b) \\ &= \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \end{aligned}$$

The integral from the first method

$$\int_a^b f(x) dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2) dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x) dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

### Example 1

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time,  $T(s)$  is given by

$$T = -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left( \frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx$$

- Use Simpson's 1/3 rule to find the time required for 50 % of the oxygen to be consumed.

- b) Find the true error,  $E_t$ , for part (a).  
 c) Find the absolute relative true error,  $|\epsilon_t|$ , for part (a).

**Solution**

$$a) \quad T \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$a = 1.22 \times 10^{-6}$$

$$b = 0.61 \times 10^{-6}$$

$$\frac{a+b}{2} = 0.91500 \times 10^{-6}$$

$$f(x) = - \left[ \frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right]$$

$$f(1.22 \times 10^{-6}) = - \left[ \frac{6.73(1.22 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} (1.22 \times 10^{-6})} \right] = -3.0581 \times 10^{11}$$

$$f(0.61 \times 10^{-6}) = - \left[ \frac{6.73(0.61 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} (0.61 \times 10^{-6})} \right] = -3.2104 \times 10^{11}$$

$$f(0.91500 \times 10^{-6}) = - \left[ \frac{6.73(0.91500 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} (0.91500 \times 10^{-6})} \right] = -3.1089 \times 10^{11}$$

$$\begin{aligned} T &= \left( \frac{b-a}{6} \right) \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= \left( \frac{0.61 \times 10^{-6} - 1.22 \times 10^{-6}}{6} \right) \left[ f(1.22 \times 10^{-6}) + 4f(0.915 \times 10^{-6}) + f(0.61 \times 10^{-6}) \right] \\ &= \left( \frac{-0.61 \times 10^{-6}}{6} \right) \left[ -3.0581 \times 10^{11} + 4(-3.1089 \times 10^{11}) - 3.2104 \times 10^{11} \right] \\ &= 190160 \text{ s} \end{aligned}$$

- b) The exact value of the above integral is,

$$T = - \int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left( \frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx$$

$$= 1.90140 \times 10^5 \text{ s}$$

so the true error is

$$\begin{aligned} E_t &= \text{True Value} - \text{Approximate Value} \\ &= 1.90140 \times 10^5 - 190160 \\ &= -24.100 \end{aligned}$$

c) The absolute relative true error,  $|\epsilon_t|$ , would then be

$$\begin{aligned} |\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{-24.020}{1.90140 \times 10^5} \right| \times 100 \\ &= 0.012675\% \end{aligned}$$

### Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval  $[a, b]$  into  $n$  segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that  $n$  needs to be even. Divide interval  $[a, b]$  into  $n$  equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x) dx = \int_{x_0}^{x_n} f(x) dx$$

where

$$x_0 = a$$

$$x_n = b$$

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-4}}^{x_{n-2}} f(x) dx + \int_{x_{n-2}}^{x_n} f(x) dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\begin{aligned} \int_a^b f(x) dx &\cong (x_2 - x_0) \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + (x_4 - x_2) \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots \\ &+ (x_{n-2} - x_{n-4}) \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + (x_n - x_{n-2}) \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right] \end{aligned}$$

Since

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \dots, n$$

then

$$\begin{aligned} \int_a^b f(x) dx &\cong 2h \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + 2h \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots \\ &+ 2h \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + 2h \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right] \end{aligned}$$



$$\begin{aligned}
 &= \frac{h}{3} [f(x_0) + 4\{f(x_1) + f(x_3) + \dots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \dots + f(x_{n-2})\} + f(x_n)] \\
 &= \frac{h}{3} \left[ f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right] \\
 \boxed{\int_a^b f(x) dx \cong \frac{b-a}{3n} \left[ f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right]}
 \end{aligned}$$

**Example 2**

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time,  $T(s)$  is given by

$$T = -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left( \frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx$$

- Use four-Simpson's 1/3 Rule to find the time required for 50% of the oxygen to be consumed.
- Find the true error,  $E_t$ , for part (a).
- Find the absolute relative true error,  $|\epsilon_t|$ , for part (a).

**Solution**

$$\begin{aligned}
 \text{a) } T &= \frac{b-a}{3n} \left[ f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right] \\
 n &= 4 \\
 a &= 1.22 \times 10^{-6} \\
 b &= 0.61 \times 10^{-6} \\
 h &= \frac{b-a}{n} \\
 &= \frac{0.61 \times 10^{-6} - 1.22 \times 10^{-6}}{4} \\
 &= -0.15250 \times 10^{-6}
 \end{aligned}$$

$$f(x) = -\left[ \frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right]$$

So

$$f(x_0) = f(1.22 \times 10^{-6})$$

$$f(1.22 \times 10^{-6}) = -\left[ \frac{6.73(1.22 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(1.22 \times 10^{-6})} \right] = -3.0581 \times 10^{11}$$

$$f(x_1) = f(1.22 \times 10^{-6} - 0.15250 \times 10^{-6}) = f(1.0675 \times 10^{-6})$$

$$f(1.0675 \times 10^{-6}) = -\left[ \frac{6.73(1.0675 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(1.0675 \times 10^{-6})} \right] = -3.0799 \times 10^{11}$$

$$f(x_2) = f(1.0675 \times 10^{-6} - 0.15250 \times 10^{-6}) = f(0.915 \times 10^{-6})$$

$$f(0.915 \times 10^{-6}) = -\left[ \frac{6.73(0.915 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(0.915 \times 10^{-6})} \right] = -3.1089 \times 10^{11}$$

$$f(x_3) = f(0.915 \times 10^{-6} - 0.15250 \times 10^{-6}) = f(0.76250 \times 10^{-6})$$

$$f(0.76250 \times 10^{-6}) = -\left[ \frac{6.73(0.76250 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(0.76250 \times 10^{-6})} \right] = -3.1495 \times 10^{11}$$

$$f(x_4) = f(x_n) = f(0.61 \times 10^{-6})$$

$$f(0.61 \times 10^{-6}) = -\left[ \frac{6.73(0.61 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(0.61 \times 10^{-6})} \right] = -3.2104 \times 10^{11}$$

$$\begin{aligned} T &= \frac{b-a}{3n} \left[ f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right] \\ &= \frac{0.61 \times 10^{-6} - 1.22 \times 10^{-6}}{3(4)} \left[ f(1.22 \times 10^{-6}) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^3 f(x_i) \right. \\ &\quad \left. + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^2 f(x_i) + f(0.61 \times 10^{-6}) \right] \\ &= \frac{-0.61 \times 10^{-6}}{12} [f(1.22 \times 10^{-6}) + 4f(x_1) + 4f(x_3) + 2f(x_2) + f(0.61 \times 10^{-6})] \end{aligned}$$

$$\begin{aligned}
&= \frac{-0.61 \times 10^{-6}}{12} \left[ f(1.22 \times 10^{-6}) + 4f(1.0675 \times 10^{-6}) + \right. \\
&\quad \left. 4f(0.76250 \times 10^{-6}) + 2f(0.915 \times 10^{-6}) + f(0.61 \times 10^{-6}) \right] \\
&= \frac{-0.61 \times 10^{-6}}{12} \left[ -3.0582 \times 10^{11} + 4(-3.0799 \times 10^{11}) + \right. \\
&\quad \left. 4(-3.1495 \times 10^{11}) + 2(-3.1089 \times 10^{11}) - 3.2104 \times 10^{11} \right] \\
&= 190140 \text{ s}
\end{aligned}$$

b) The exact value of the above integral is

$$\begin{aligned}
T &= -\int_{1.22 \times 10^{-6}}^{0.61 \times 10^{-6}} \left( \frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11} x} \right) dx \\
&= 1.90140 \times 10^5 \text{ s}
\end{aligned}$$

so the true error is

$$\begin{aligned}
E_t &= \text{True Value} - \text{Approximate Value} \\
&= 1.90140 \times 10^5 - 190140 \\
&= -1.9838
\end{aligned}$$

c) The absolute relative true error,  $|\epsilon_t|$ , would then be

$$\begin{aligned}
|\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\
&= \left| \frac{-1.9838}{1.90140 \times 10^5} \right| \times 100 \\
&= 0.0010434 \%
\end{aligned}$$

**Table 1** Values of Simpson's 1/3 Rule for Example 2 with multiple segments.

$n$	Approximate Value	$E_t$	$ \epsilon_t $ %
2	190160	-24.100	0.012675
4	190140	-1.9838	0.0010434
6	190140	-0.42010	$2.2094 \times 10^{-4}$
8	190140	-0.13655	$7.1815 \times 10^{-5}$
10	190140	-0.056663	$2.9802 \times 10^{-5}$

### Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given<sup>1</sup> by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the  $n$  segments Simpson's 1/3rd Rule is given by

$$\begin{aligned} E_1 &= -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \\ &= -\frac{h^5}{90} f^{(4)}(\zeta_1) \\ E_2 &= -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \\ &= -\frac{h^5}{90} f^{(4)}(\zeta_2) \\ &\vdots \\ E_i &= -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i} \\ &= -\frac{h^5}{90} f^{(4)}(\zeta_i) \\ &\vdots \\ E_{\frac{n}{2}-1} &= -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2} \\ &= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right) \\ E_{\frac{n}{2}} &= -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n \end{aligned}$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$\begin{aligned} &= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right) \\ E_t &= \sum_{i=1}^{\frac{n}{2}} E_i \end{aligned}$$

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<sup>1</sup> The  $f^{(4)}$  in the true error expression stands for the fourth derivative of the function  $f(x)$ .

$$\begin{aligned}
 &= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \\
 &= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \\
 &= -\frac{(b-a)^5}{90n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}
 \end{aligned}$$

The term  $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$  is an approximate average value of  $f^{(4)}(x)$ ,  $a < x < b$ . Hence

$$E_t = -\frac{(b-a)^5}{90n^4} \bar{f}^{(4)}$$

where

$$\bar{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

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#### INTEGRATION

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Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	Chemical Engineering
Authors	Autar Kaw, Michael Keteltas
Date	November 8, 2012
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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