Runge 2nd Order Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

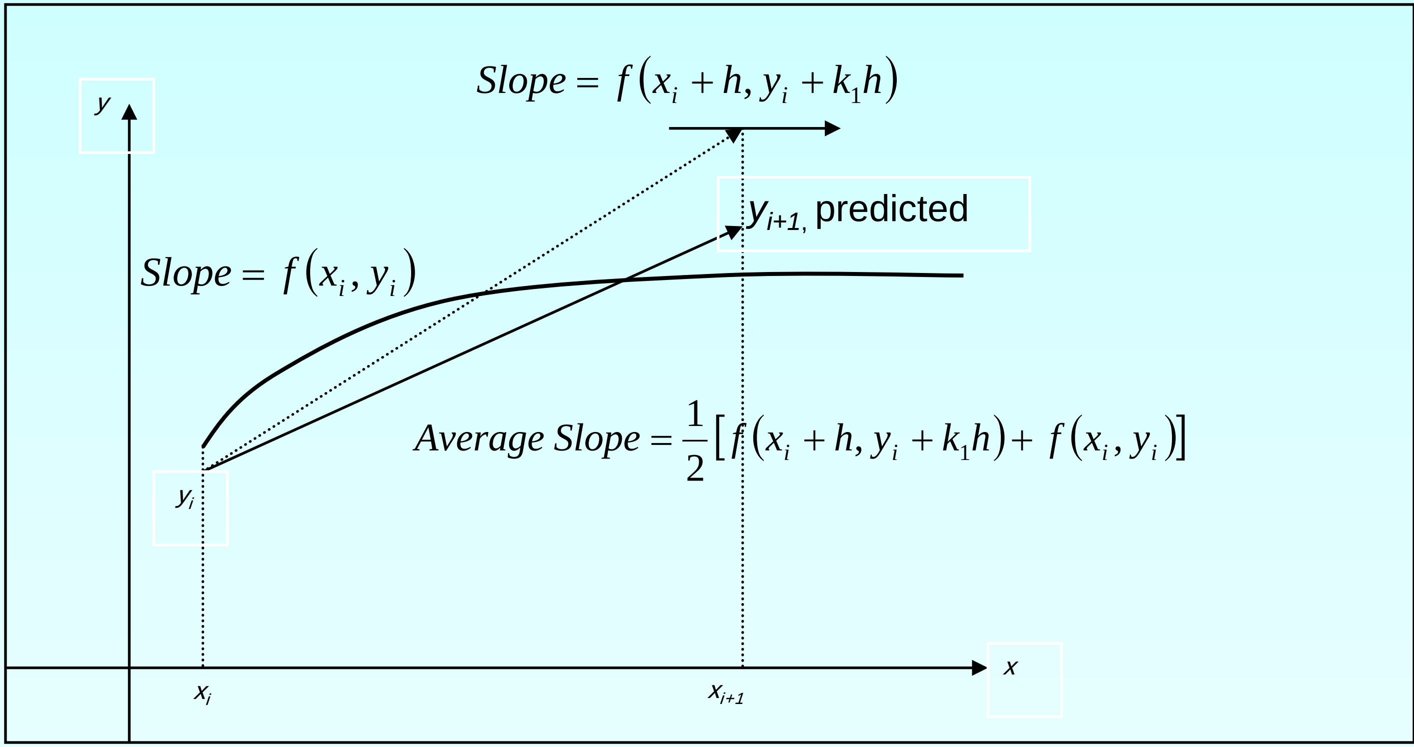

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The concentration of salt, x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, t = 0, the salt concentration in the tank is 50g/L. Using Euler's method and a step size of h=1.5 min, what is the salt concentration after 3 minutes.

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t, x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0,\ t_0 = 0,\ x_0 = 50$

$$k_1 = f(t_0, x_o) = f(0,50) = 37.5 - 3.5(50) = -137.50$$

$$k_2 = f(t_0 + h, x_0 + k_1 h) = f(0 + 1.5, 50 + (-137.50)1.5) = f(1.5, -156.25)$$
$$= 37.5 - 3.5(-156.25) = 584.38$$

$$x_1 = x_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 50 + \left(\frac{1}{2}(-137.50) + \frac{1}{2}(584.38)\right)1.5$$
$$= 50 + (223.44)1.5$$
$$= 385.16 g / L$$

x_1 is the approximate concentration of salt at $t = t_1 = t_0 + h = 0 + 1.5 = 1.5\,\text{min}$

$$x(1.5) \approx x_1 = 385.16\,\text{g/L}$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 1.5 = 1.5, x_1 = 385.16 g/L$

$$k_1 = f(t_1, x_1) = f(1.5, 385.16) = 37.5 - 3.5(385.16) = -1310.5$$

$$k_2 = f(t_1 + h, x_1 + k_1 h) = f(1.5 + 1.5, 385.16 + (-1310.5)1.5) = f(3, -1580.6)$$
$$= 37.5 - 3.5(-1580.6) = 5569.8$$

$$x_2 = x_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 385.16 + \left(\frac{1}{2}(-1310.5) + \frac{1}{2}(5569.8)\right)1.5$$

$$= 385.16 + (2129.6)1.5$$

$$= 3579.7 g/L$$

x_1 is the approximate concentration of salt at $t = t_2 = t_1 + h = 1.5 + 1.5 = 3\,\text{min}$

$$x(3) \approx x_1 = 3579.7\text{ g/L}$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5x}$$

The solution to this nonlinear equation at t=3 minutes is

$$x(3) = 10.715 \text{ g/L}$$

Comparison with exact results

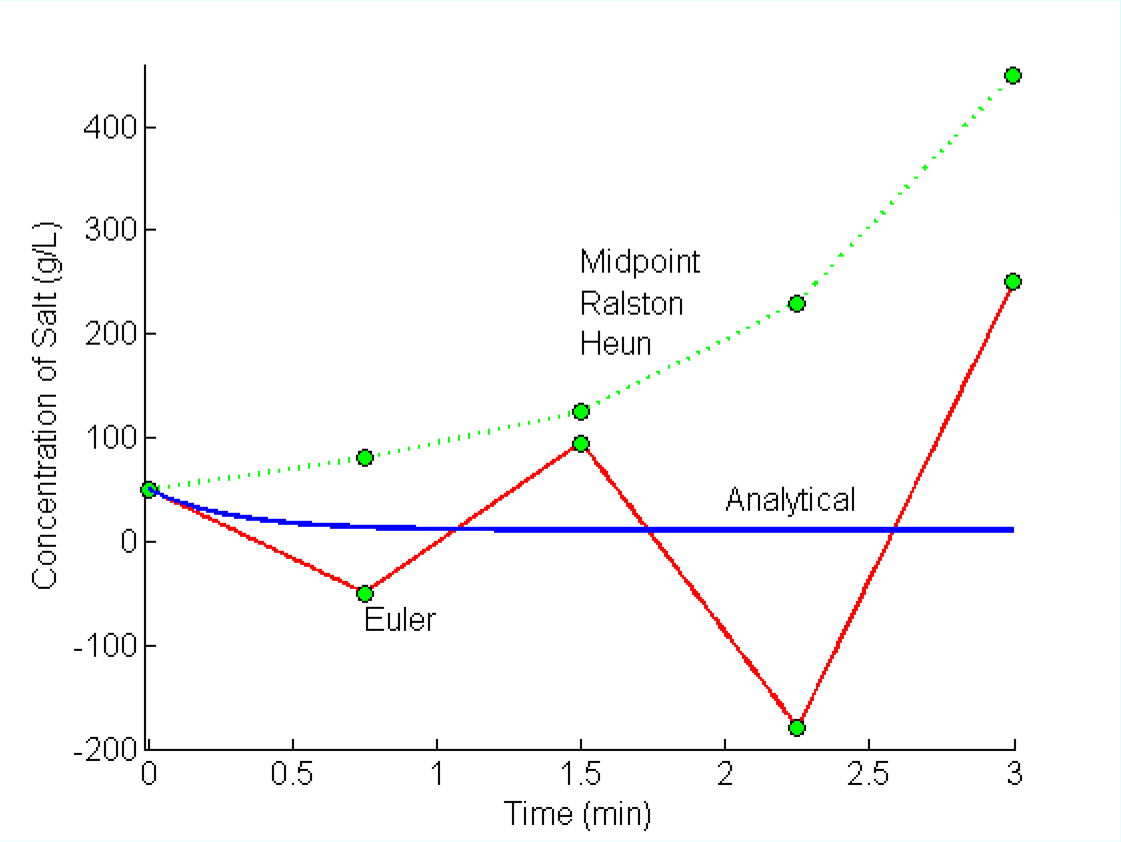

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Effect of step size for Heun's method

Step size, h	$x(3)$	E_t	$\lvert\in_t\rvert\ \%$
3	1803.1	−1792.4	16727
1.5	3579.6	−3568.9	33306
0.75	442.05	−431.34	4025.4
0.375	11.038	−0.32231	3.0079
0.1875	10.718	−0.0024979	0.023311

$x(3) = 10.715$ (exact)

Effects of step size on Heun's Method

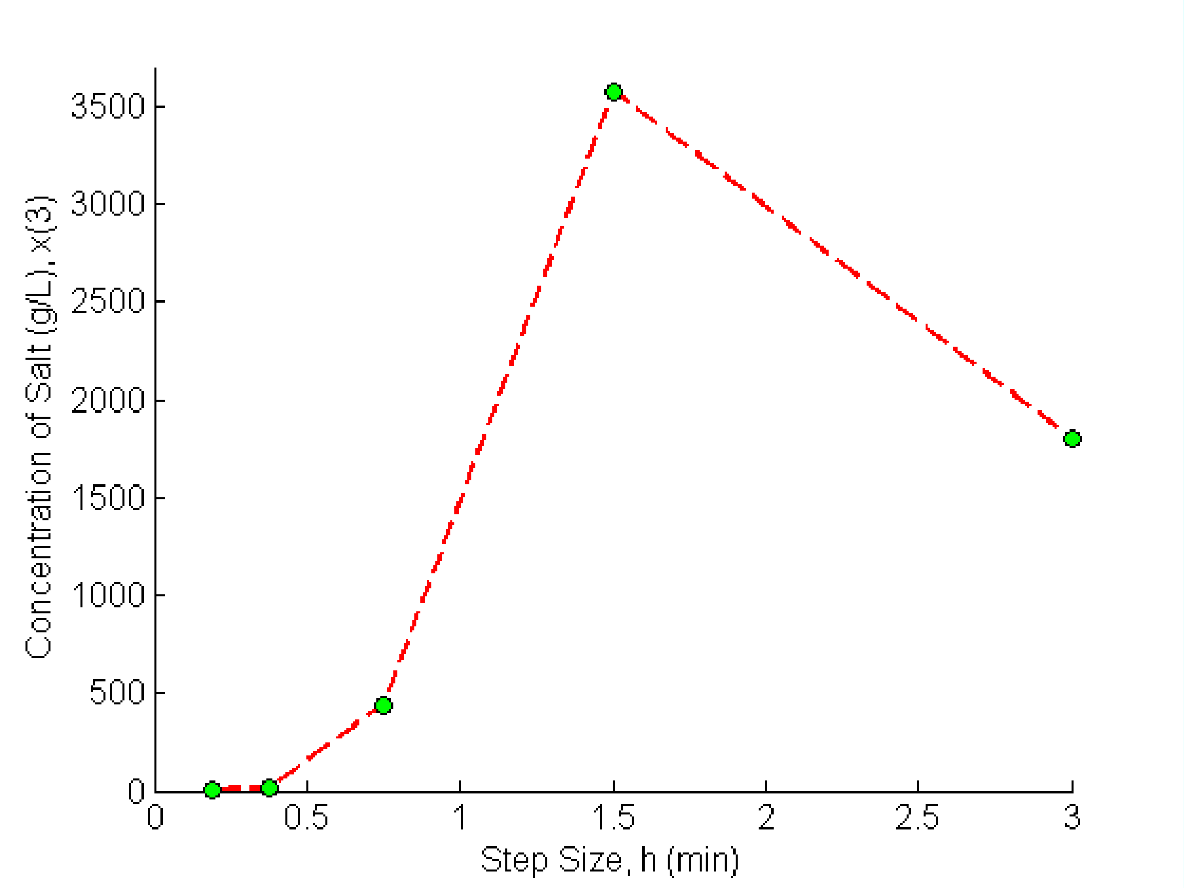

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2^{nd} Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$x(3)$			
	Euler	Heun	Midpoint	Ralston
3	−362.50	1803.1	1803.1	1803.1
1.5	720.31	3579.6	3579.6	3579.6
0.75	284.65	442.05	442.05	442.05
0.375	10.718	11.038	11.038	11.038
0.1875	10.714	10.718	10.718	10.718

$x(3) = 10.715$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\left\|\in_t\right\|\%$			
	Euler	Heun	Midpoint	Ralston
3	3483.0	16727	16727	16727
1.5	6622.2	33306	33306	33306
0.75	2556.5	4025.4	4025.4	4025.4
0.375	0.023249	3.0079	3.0079	3.0079
0.1875	0.010082	0.023311	0.023311	0.023311

$x(3) = 10.715$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

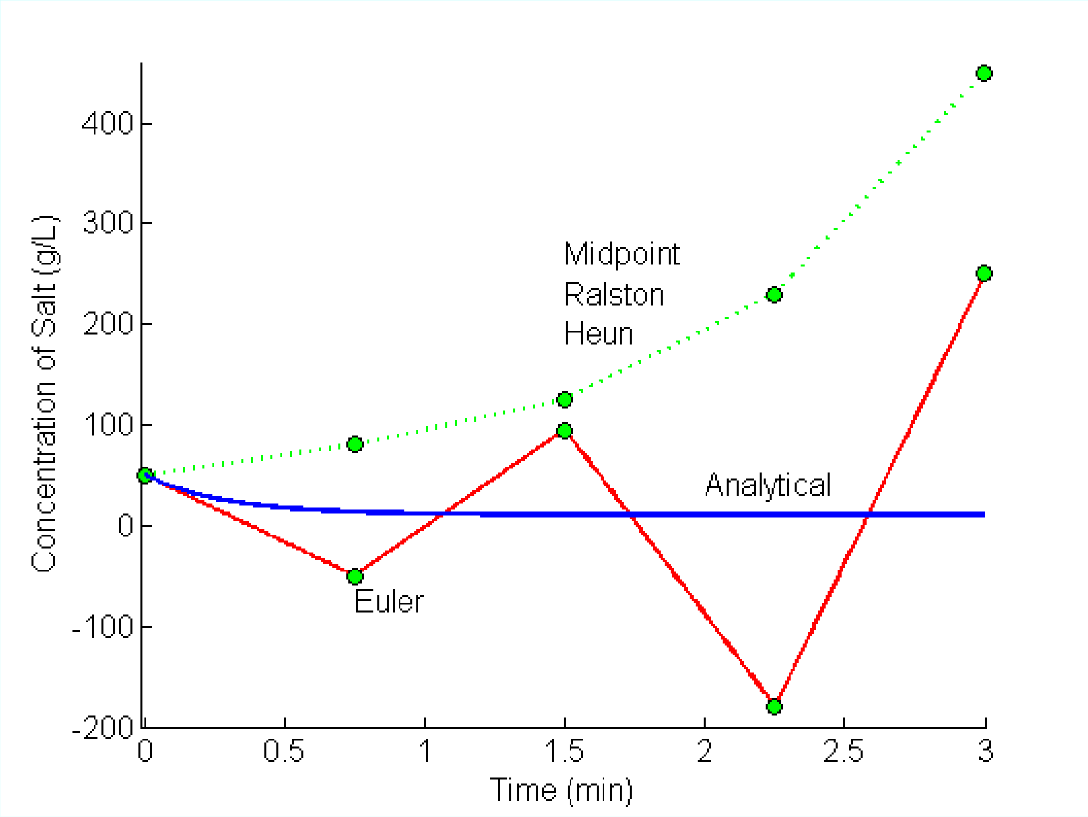

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

THE END

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