Runge 4th Order Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 4th Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The concentration of salt, x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, t = 0, the salt concentration in the tank is 50g/L. Using Euler's method and a step size of h=1.5 min, what is the salt concentration after 3 minutes.

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t, x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: $i=0,\ \ t_0=0,\ \ x_0=50g/L$

$$k_1 = f\left(t_0,\ x_0\right) = f\left(0,50\right) = 37.5 - 3.5\left(50\right) = -137.5$$

$$k_2 = f\left(t_0+\frac{1}{2}h,\ x_0+\frac{1}{2}k_1h\right) = f\left(0+\frac{1}{2}\left(1.5\right),\ 50+\frac{1}{2}\left(-137.5\right)1.5\right)$$

$$= f\left(0.75,\ -53.125\right) = 37.5 - 3.5\left(-53.125\right) = 223.44$$

$$k_3 = f\left(t_0+\frac{1}{2}k_2h\right) = f\left(0+\frac{1}{2}\left(1.5\right),\ 50+\frac{1}{2}\left(223.44\right)1.5\right)$$

$$= f\left(0.75,\ 217.58\right) = 37.5 - 3.5\left(217.58\right) = -724.02$$

$$k_4 = \left(t_0+h,\ x_0+k_3h\right) = f\left(0+1.5,\ 50+\left(-724.03\right)1.5\right)$$

$$= f\left(1.5,\ -1036.0\right) = 37.5 - 3.5\left(1036.0\right) = 3663.6$$

Solution Cont

$$x_1 = x_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 50 + \frac{1}{6}\left(-137.5 + 2\left(223.44\right) + 2\left(-724.02\right) + \left(3663.6\right)\right)1.5$$

$$= 50 + \frac{1}{6}\left(2525.0\right)1.5$$

$$= 681.24 \text{ g/L}$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5$$

$$x(1.5) \approx x_1 = 681.24\, g/L$$

Solution Cont

Step 2: $i=1,\ t_1=1.5,\ x_1=681.24\ g/L$

$$k_1 = f(t_1,\ x_1) = f(1.5,\ 681.24) = 37.5 - 3.5(681.24) = -2346.8$$

$$k_2 = f\left(t_1+\frac{1}{2}h, x_1+\frac{1}{2}k_1h\right) = f\left(1.5+\frac{1}{2}1.5, 681.24+\frac{1}{2}(-2346.8)1.5\right)$$
$$= f(2.25,-1078.9) = 37.5-3.5(-1078.9) = 38.13.6$$

$$k_3 = f\left(t_1+\frac{1}{2}h, x_1+\frac{1}{2}k_2h\right) = f\left(1.5+\frac{1}{2}1.5,\ 681.24+\frac{1}{2}(3813.6)1.5\right)$$
$$= f(2.25,\ 3541.4) = 37.5-3.5(3541.4) = -12358$$

$$k_4 = f(t_1+h, x_1+k_3h) = f(1.5+1.5,\ 681.24+(-12358)1.5)$$
$$= f(3,-17855) = 37.5-3.5(-17855) = 62530$$

Solution Cont

$$x_2 = x_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 681.24 + \frac{1}{6}\left(-2346.8 + 2\left(3813.6\right) + 2\left(-12358\right) + 62530\right)1.5$$

$$= 681.24 + \frac{1}{6}\left(43096\right)1.5$$

$$= 11455 \text{ g/L}$$

x_2 is the approximate concentration of salt at

$$t_2 = t_1 + h = 1.5 + 1.5 = 3\,\text{min}.$$

$$x(3) \approx x_2 = 11455 g / L$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at t=3 minutes is

$$x(3) = 10.715$$

Comparison with exact results

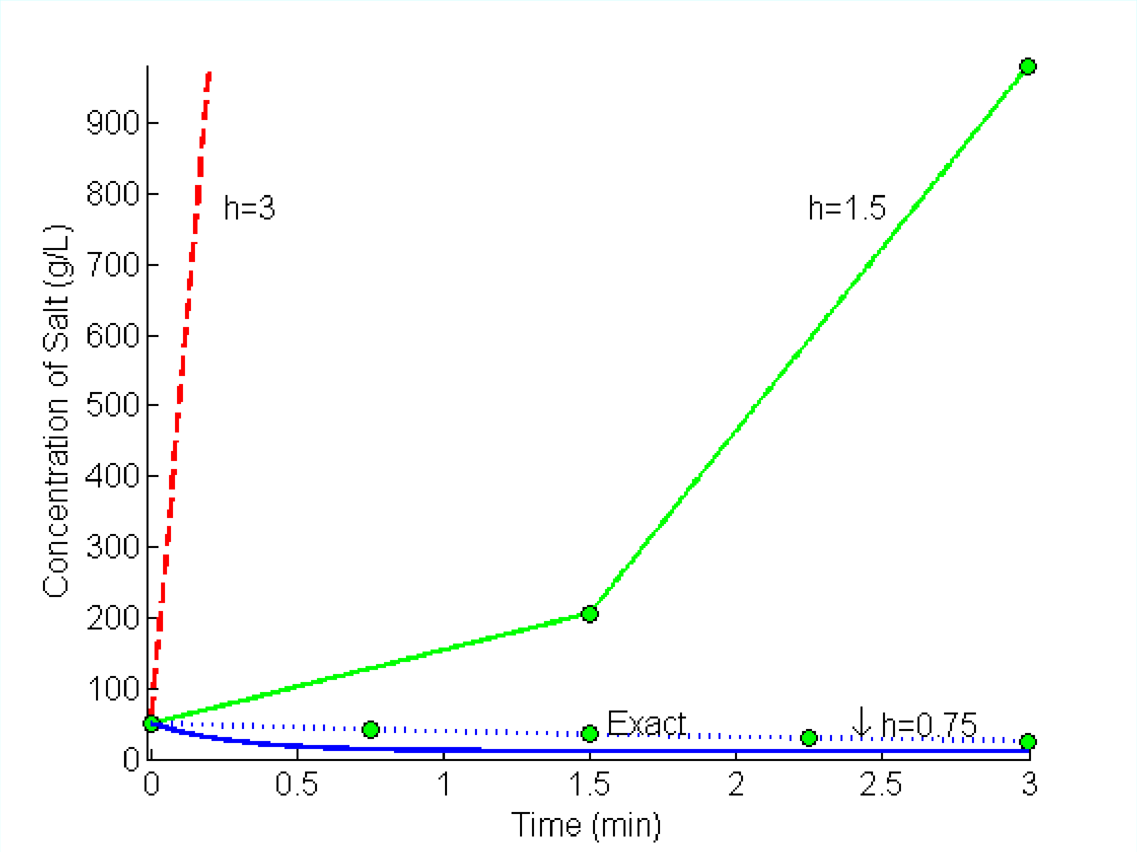

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1 Value of concentration of salt at 3 minutes for different step sizes

Step size, h	$x(3)$	E_t	$\lvert \in_t \rvert \%$
3	14120	−14109	131680
1.5	11455	−11444	106800
0.75	25.559	−14.843	138.53
0.375	10.717	−0.0014969	0.013969
0.1875	10.715	−0.00031657	0.0029544

$x(3) = 10.715$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

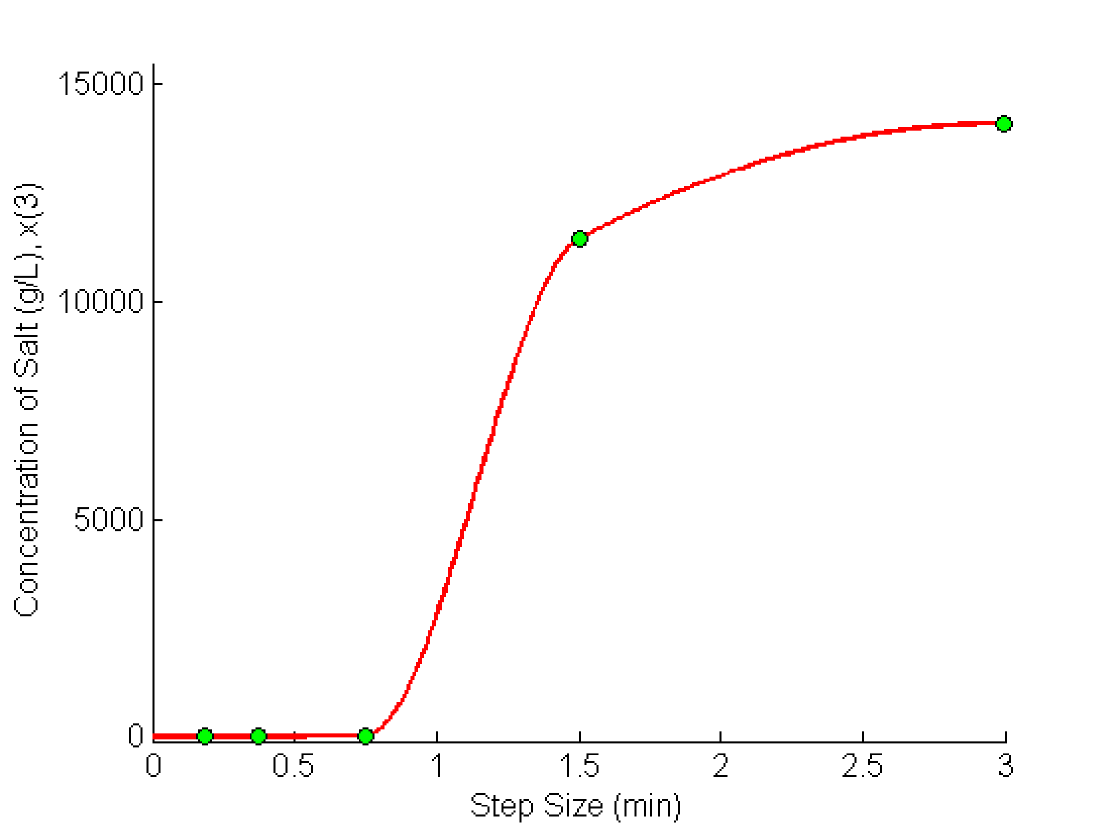

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

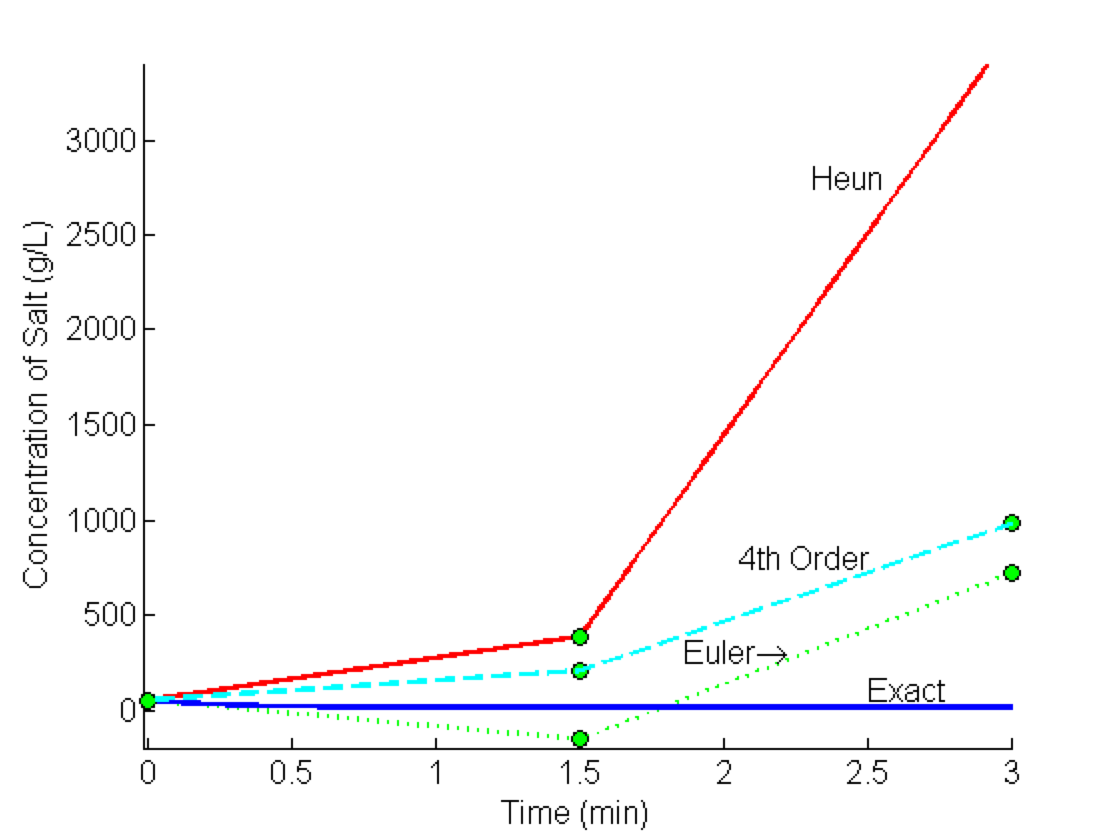

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

THE END

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