Chapter 04.00C

Physical Problem for Simultaneous Linear Equations
Civil Engineering

Problem Statement: A pressure vessel can only be subjected to an amount of internal pressure that is limited by the strength of material used. For example, take a pressure vessel of internal radius of $a = 5''$ and outer radius, $b = 8''$, made of ASTM 36 steel (yield strength of ASTM 36 steel is 36 ksi). How much internal pressure can this pressure vessel take before it is considered to have failed?

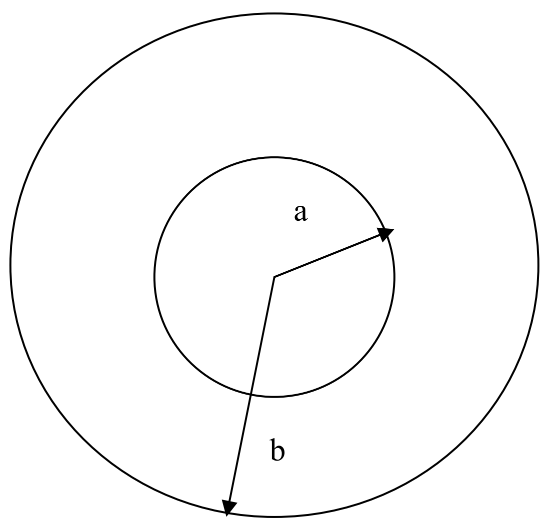

Figure 1 A single cylinder pressure vessel with internal radius, a and outer radius, b.

The hoop and radial stress in a cylindrical pressure vessel is given by [1]

$$\sigma_r = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \tag{1}$$

$$\sigma_\theta = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{2}$$

The maximum normal stress anywhere in the cylinder is the hoop stress at the inner radius, a

$$\sigma_\theta\Big|_{\max} = p_i\left(\frac{b^2+a^2}{b^2-a^2}\right) \tag{3}$$

Assuming a factor of safety of 2, while the yield strength is given as 36 ksi,

$$\frac{36\times10^3}{2} = p_i\left(\frac{8^2+5^2}{8^2-5^2}\right)$$

$$p_i = 7.887\,\text{ksi}$$

You can see from Equation (3) that even for $b >> a$, the maximum internal pressure one can apply is only $p_i = 18\,\text{ksi}$. Therefore, what can an engineer do to maximize the internal pressure, while keeping the material and radial dimensions the same? He or she can use a compounded cylinder. One can create a compounded cylinder by shrink fitting one cylinder into another, and hence creating pre-existing favorable stresses to allow more internal pressure. Let us see how that would work?

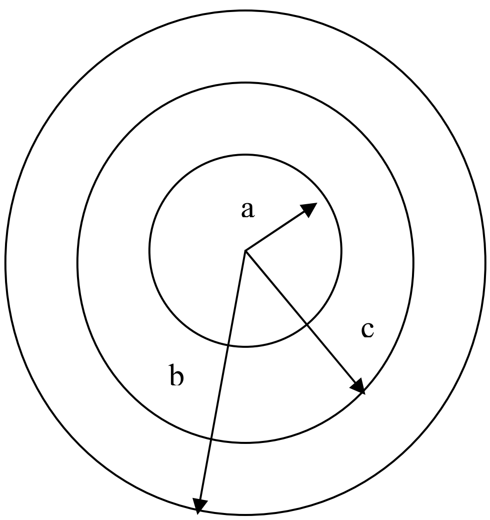

Figure 2 A compounded cylinder pressure vessel with internal radius, a, outer radius, b, and interface at $r = c$.

Let us make the compounded cylinder of two cylinders (Figure 2). Cylinder 1 has an internal radius of $a = 5"$, and outer radius $c = 6.5"$, while Cylinder 2 has an internal radius of $c = 6.5"$ and outer radius, $b = 8"$. Assume that that radial interference, $\delta = 0.007"$ occurs at the interface of a compounded cylinder at $r = c = 6.5"$. How does one find then the pressure that can be applied to the compounded cylinder of internal radius, $a = 5"$ and outer radius, $b = 8"$?

For a cylinder 1, the radial displacement, u_1 is given by

$$u_1 = c_1 r + \frac{c_2}{r} \tag{4}$$

the radial stress, σ_r^1 and hoop stress, σ_θ^1 by

$$\sigma_r^1 = \frac{E}{1-\nu^2}\left[c_1(1+\nu) - c_2\left(\frac{1-\nu}{r^2}\right)\right] \tag{5}$$

$$\sigma_\theta^1 = \frac{E}{1-\nu^2}\left[c_1(1+\nu) + c_2\left(\frac{1-\nu}{r^2}\right)\right] \tag{6}$$

where

E = Young's modulus of steel,
ν = Poisson's ratio of steel.

For cylinder 2, the radial displacements, u_2 is given by

$$u_2 = c_3 r + \frac{c_4}{r} \tag{7}$$

the radial stress, σ_r^2 and hoop stress, σ_θ^2 by

$$\sigma_r^2 = \frac{E}{1-\nu^2}\left[c_3(1+\nu) - c_4\left(\frac{1-\nu}{r^2}\right)\right] \tag{8}$$

$$\sigma_\theta^2 = \frac{E}{1-\nu^2}\left[c_3(1+\nu) + c_4\left(\frac{1-\nu}{r^2}\right)\right] \tag{9}$$

So if one is able to find the four constants, c_1, c_2, c_3 and c_4, one can find the stresses in the compounded cylinder to be able to find what internal pressure can be applied. So how do we find the four unknown constants?

The boundary and interface conditions are

The radial stress at the inner radius, $r = a$ is the applied internal pressure

$$\sigma_r^1(r=a) = -p_i \tag{10}$$

The radial stress is continuous at the interface, $r = c$

$$\sigma_r^1(r=c) = \sigma_r^2(r=c) \tag{11}$$

The radial displacement at the interface, $r = c$ has a jump of the radial interference, δ

$$u_2(r=c) - u_1(r=c) = \delta \tag{12}$$

The radial stress at the outer radius, $r = c$ is

$$\sigma_r^2(r=b) = 0 \tag{13}$$

This will set up four equations and four unknowns, if we know what internal pressure we are applying. Assume, we are applying the same pressure as the single cylinder can take, that is, $p_i = 7.887$ ksi and let us see later what stresses it creates in the compounded cylinder.

Assuming $E = 30 \times 10^6$ psi, $\nu = 0.3$, Equations (10) through (13) become

$$\frac{30\times10^6}{1-0.3^2}\left[c_1(1+0.3) - c_2\left(\frac{1-0.3}{5^2}\right)\right] = -7.887\times10^3$$

$$\frac{30\times10^6}{1-0.3^2}\left[c_1(1+0.3) - c_2\left(\frac{1-0.3}{6.5^2}\right)\right] = \frac{30\times10^6}{1-0.3^2}\left[c_3(1+0.3) - c_4\left(\frac{1-0.3}{6.5^2}\right)\right]$$

$$c_3(6.5)+\frac{c_4}{6.5}-c_1(6.5)-\frac{c_2}{6.5}=0.007$$

$$\frac{30\times10^6}{1-0.3^2}\left[c_3(1+0.3)-c_4\left(\frac{1-0.3}{8^2}\right)\right]=0$$

Writing the above equations in matrix form, we get

$$\begin{bmatrix} 4.2857\times10^7 & -9.2307\times10^5 & 0 & 0 \\ 4.2857\times10^7 & -5.4619\times10^5 & -4.2857\times10^7 & 5.4619\times10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^7 & -3.6057\times10^5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}=\begin{bmatrix} -7.887\times10^3 \\ 0 \\ 0.007 \\ 0 \end{bmatrix} \tag{14}$$

Solving these four simultaneous linear equations, we can find the four constants.

REFERENCES

1. A.C. Ugural, S.K. Fenster, Advanced strength and applied elasticity, Third Edition, Prentice Hall, New York, 1995.
2. J.E. Shigley, C.R. Mischke, Chapter 19 - Limits and fits, Standard handbook of machine design, McGraw-Hill, New York, 1986.

QUESTIONS

1. Find the unknown constants of Equation (14) using different numerical methods.
2. Knowing that the critical points in the compounded cylinder are $r = a, c-, c+, \text{and}\, b$, find the maximum hoop stress in the compounded cylinder. What is its value compared to the maximum hoop stress allowable of 18 ksi?
3. Find the maximum internal pressure you can apply to the compounded cylinder? Compare it with the maximum possible internal pressure for a single cylinder of same dimensions.
4. The radial interference at the interface is created by making the inner cylinder 1 to have a larger outer radius than the inner radius of cylinder 2. Standard interference fits dictate the limits of these dimensions. If a cylinder 2 is fit into cylinder 1, there is an upper and lower limit by which the nominal diameter of each cylinder varies at the interface. This limit L in thousands of an inch, is given by [2]

   $$L = CD^{1/3}$$

   where D (nominal diameter) is in inches and the coefficient C, based on the type of fit, is given in Table 1 below.

Cylinder	Limit	Class of fit	
		FN2	FN3
1	Lower	0.000	0.000
	Upper	0.907	0.907
2	Lower	2.717	3.739
	Upper	3.288	4.310

Assuming FN2 fit at the interface, find the maximum internal pressure you would recommend.

SIMULATENAOUS LINEAR EQUATIONS	
Topic	Simultaneous Linear Equations
Summary	Compounded cylinder capability vs single cylinder capability to handle internal pressures
Major	Civil Engineering
Authors	Autar Kaw
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