

# Gaussian Elimination

Civil Engineering Majors

Author(s): Autar Kaw

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Transforming Numerical Methods Education for STEM  
Undergraduates

# Naïve Gauss Elimination

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# Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form  $[A][X]=[C]$

Two steps

1. Forward Elimination
2. Back Substitution

# Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$



$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

# Forward Elimination

A set of  $n$  equations and  $n$  unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

( $n-1$ ) steps of forward elimination

# Forward Elimination

## Step 1

For Equation 2, divide Equation 1 by  $a_{11}$  and multiply by  $a_{21}$ .

$$\left[ \begin{array}{c} a_{21} \\ a_{11} \end{array} \right] (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

# Forward Elimination

Subtract the result from Equation 2.

$$\begin{array}{r} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ - \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1 \\ \hline \left( a_{22} - \frac{a_{21}}{a_{11}}a_{12} \right) x_2 + \dots + \left( a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \right) x_n = b_2 - \frac{a_{21}}{a_{11}}b_1 \end{array}$$

$$\text{or} \quad a'_{22}x_2 + \dots + a'_{2n}x_n = b'_2$$

# Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \dots + a'_{3n}x_n = b'_3$$

$$\vdots \quad \vdots \quad \vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \dots + a'_{nn}x_n = b'_n$$

**End of Step 1**



# Forward Elimination

## Step 2

Repeat the same procedure for the 3<sup>rd</sup> term of Equation 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{3n}x_n = b''_3$$

$$\vdots \quad \vdots$$

$$a''_{n3}x_3 + \dots + a''_{nn}x_n = b''_n$$

**End of Step 2**

# Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{3n}x_n = b''_3$$

$$\vdots$$

$$a^{(n-1)}_{nn}x_n = b^{(n-1)}_n$$

**End of Step (n-1)**

# Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a^{(n-1)}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b^{(n-1)}_n \end{bmatrix}$$

# Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

# Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_n x_n = b''_3$$

$$\vdots$$

$$a^{(n-1)}_{nn}x_n = b^{(n-1)}_n$$

# Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

# Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)}x_{i+1} - a_{i,i+2}^{(i-1)}x_{i+2} - \dots - a_{i,n}^{(i-1)}x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \dots, 1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \dots, 1$$

**THE END**

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# Naïve Gauss Elimination Example

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# Example: Cylinder Stresses

To find the maximum stresses in a compound cylinder, the following four simultaneous linear equations need to be solved.

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 4.2857 \times 10^7 & -5.4619 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

In the compound cylinder, the inner cylinder has an internal radius of  $a = 5''$  and an outer radius  $c = 6.5''$ , while the outer cylinder has an internal radius of  $c = 6.5''$  and outer radius of  $b = 8''$ . Given  $E = 30 \times 10^6$  psi,  $\nu = 0.3$ , and that the hoop stress in outer cylinder is given by

$$\sigma_{\theta} = \frac{E}{1-\nu^2} \left[ c_3(1+\nu) + c_4 \left( \frac{1-\nu}{r^2} \right) \right]$$

find the stress on the inside radius of the outer cylinder.

Find the values of  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$  using Naïve Gauss Elimination.

# Example: Cylinder Stresses

Forward Elimination: Step 1

$$\text{Row2} - \left[ \frac{4.2857 \times 10^7}{4.2857 \times 10^7} \right] \times (\text{Row1}) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 0.007 \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 1

$$\text{Row3} - \left[ \frac{-6.5}{4.2857 \times 10^7} \right] \times (\text{Row1}) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 5.8038 \times 10^{-3} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 1

$$\text{Row4} - \left[ \frac{0}{4.2857 \times 10^7} \right] \times (\text{Row1}) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & -0.29384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 5.8038 \times 10^{-3} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 2

$$\text{Row3} - \left[ \frac{-0.29384}{3.7688 \times 10^5} \right] \times (\text{Row2}) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 1.1953 \times 10^{-2} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 2

$$\text{Row4} - \left[ \frac{0}{3.7688 \times 10^5} \right] \times (\text{Row2}) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 1.1953 \times 10^{-2} \\ 0 \end{bmatrix}$$



# Example: Cylinder Stresses

Forward Elimination: Step 3

$$\text{Row4} - \left[ \frac{4.2857 \times 10^7}{-26.914} \right] \times (\text{Row3}) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & 0 & -26.914 & 0.57968 \\ 0 & 0 & 0 & 5.625 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 1.1953 \times 10^{-2} \\ 1.9034 \times 10^4 \end{bmatrix}$$

This is now ready for Back Substitution.

# Example: Cylinder Stresses

Back Substitution: Solve for  $c_4$  using the fourth equation

$$\begin{aligned}5.625 \times 10^5 c_4 &= 1.9034 \times 10^4 \\c_4 &= \frac{1.9034 \times 10^4}{5.625 \times 10^5} \\&= 3.3837 \times 10^{-2}\end{aligned}$$

# Example: Cylinder Stresses

Back Substitution: Solve for  $c_3$  using the third equation

$$-26.914c_3 + (0.57968)c_4 = 1.1953 \times 10^{-2}$$

$$c_3 = \frac{1.1953 \times 10^{-2} - (0.57968)c_4}{-26.914}$$

$$= \frac{1.1953 \times 10^{-2} - (0.57968) \times 3.3837 \times 10^{-2}}{-26.914}$$

$$= 2.8469 \times 10^{-4}$$

# Example: Cylinder Stresses

Back Substitution: Solve for  $c_2$  using the second equation

$$3.7688 \times 10^5 c_2 + (-4.2857 \times 10^7) c_3 + 5.4619 \times 10^5 c_4 = 7.887 \times 10^3$$

$$\begin{aligned} c_2 &= \frac{7.887 \times 10^3 - (-4.2857 \times 10^7) c_3 - 5.4619 \times 10^5 c_4}{3.7688 \times 10^5} \\ &= \frac{7.887 \times 10^3 - (-4.2857 \times 10^7) \times (2.8469 \times 10^{-4}) - (5.4619 \times 10^5) \times (3.3837 \times 10^{-2})}{3.7688 \times 10^5} \\ &= 4.2615 \times 10^{-3} \end{aligned}$$

# Example: Cylinder Stresses

Back Substitution: Solve for  $c_1$  using the first equation

$$4.2857 \times 10^7 c_1 + -9.2307 \times 10^5 c_2 + (0)c_3 + (0)c_4 = -7.887 \times 10^3$$

$$c_1 = \frac{-7.887 \times 10^3 - (-9.2307 \times 10^5)c_2 - (0)c_3 - (0)c_4}{4.2857 \times 10^7}$$

$$= \frac{-7.887 \times 10^3 - (-9.2307 \times 10^5) \times (4.2615 \times 10^{-3})}{4.2857 \times 10^7}$$

$$= -9.2244 \times 10^{-5}$$

# Example: Cylinder Stresses

The solution vector is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -9.2244 \times 10^{-5} \\ 4.2615 \times 10^{-3} \\ 2.8469 \times 10^{-4} \\ 3.3837 \times 10^{-2} \end{bmatrix}$$

The stress on the inside radius of the outer cylinder is then given by

$$\begin{aligned} \sigma_{\theta} &= \frac{E}{1-\nu^2} \left[ c_3(1+\nu) + c_4 \left( \frac{1-\nu}{r^2} \right) \right] \\ &= \frac{30 \times 10^6}{1-0.3^2} \left[ 2.8469 \times 10^{-4} (1+0.3) + 3.3837 \times 10^{-2} \left( \frac{1-0.3}{6.5^2} \right) \right] \\ &= 30683 \text{ psi} \end{aligned}$$

**THE END**

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# Naïve Gauss Elimination Pitfalls

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# Pitfall#1. Division by zero

$$10x_2 - 7x_3 = 3$$

$$6x_1 + 2x_2 + 3x_3 = 11$$

$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

$$12x_1 + 10x_2 - 7x_3 = 15$$

$$6x_1 + 5x_2 + 3x_3 = 14$$

$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

YES

$$12x_1 + 10x_2 - 7x_3 = 15$$

$$6x_1 + 5x_2 + 3x_3 = 14$$

$$24x_1 - x_2 + 5x_3 = 28$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix} \longrightarrow \begin{bmatrix} 12 & 10 & -7 \\ 0 & 0 & 6.5 \\ 12 & -21 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 6.5 \\ -2 \end{bmatrix}$$

Division by zero is a possibility at any step  
of forward elimination

# Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

# Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using **6** significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

# Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using **5** significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Is there a way to reduce the round off error?

# Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

# Avoiding Pitfalls

## Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error



**THE END**

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# Gauss Elimination with Partial Pivoting

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# Pitfalls of Naïve Gauss Elimination

- Possible division by zero
- Large round-off errors

# Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

# Avoiding Pitfalls

## Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

# What is Different About Partial Pivoting?

At the beginning of the  $k^{\text{th}}$  step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \dots, |a_{nk}|$$

If the maximum of the values is  $|a_{pk}|$

in the  $p^{\text{th}}$  row,  $k \leq p \leq n$ , then switch rows  $p$  and  $k$ .

# Matrix Form at Beginning of 2<sup>nd</sup> Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

## Example (2<sup>nd</sup> step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -7 & 6 & 1 & 2 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -17 & 12 & 11 & 43 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 8 \\ 9 \\ 3 \end{bmatrix}$$

Which two rows would you switch?



## Example (2<sup>nd</sup> step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$$

Switched Rows

# Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form  $[A][X]=[C]$

Two steps

1. Forward Elimination
2. Back Substitution

# Forward Elimination

Same as naïve Gauss elimination method except that we switch rows before **each** of the  $(n-1)$  steps of forward elimination.

# Example: Matrix Form at Beginning of 2<sup>nd</sup> Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

# Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a^{(n-1)}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b^{(n-1)}_n \end{bmatrix}$$

# Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{nn}x_n = b''_3$$

$$\vdots \quad \vdots$$

$$a^{(n-1)}_{nn}x_n = b^{(n-1)}_n$$

# Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n - 1, \dots, 1$$

**THE END**

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# Gauss Elimination with Partial Pivoting Example

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## Example 2

Solve the following set of equations by Gaussian elimination with partial pivoting

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

## Example 2 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

# Forward Elimination

# Number of Steps of Forward Elimination

Number of steps of forward elimination is  
 $(n-1) = (3-1) = 2$

# Forward Elimination: Step 1

- Examine absolute values of first column, first row and below.

$$|25|, |64|, |144|$$

- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

# Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 64,  $\frac{64}{144} = 0.4444$ .

$$[144 \ 12 \ 1 \ \vdots \ 279.2] \times 0.4444 = [63.99 \ 5.333 \ 0.4444 \ \vdots \ 124.1]$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ - \begin{bmatrix} 63.99 & 5.333 & 0.4444 & \vdots & 124.1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

# Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \quad \begin{array}{l} \text{Divide Equation 1 by 144 and} \\ \text{multiply it by 25, } \frac{25}{144} = 0.1736. \end{array}$$

$$[144 \ 12 \ 1 \ \vdots \ 279.2] \times 0.1736 = [25.00 \ 2.083 \ 0.1736 \ \vdots \ 48.47]$$

Subtract the result from  
Equation 3

$$\begin{array}{r} \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \\ - \begin{bmatrix} 25 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \end{array}$$

Substitute new equation for  
Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$



# Forward Elimination: Step 2

- Examine absolute values of second column, second row and below.

$$|2.667|, |2.917|$$

- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

# Forward Elimination: Step 2 (cont.)

$$\left[ \begin{array}{ccc|c} 144 & 12 & 1 & 279.2 \\ 0 & 2.917 & 0.8264 & 58.33 \\ 0 & 2.667 & 0.5556 & 53.10 \end{array} \right]$$

Divide Equation 2 by 2.917 and multiply it by 2.667,  
 $\frac{2.667}{2.917} = 0.9143$ .

$$[0 \quad 2.917 \quad 0.8264 \quad : \quad 58.33] \times 0.9143 = [0 \quad 2.667 \quad 0.7556 \quad : \quad 53.33]$$

Subtract the result from  
Equation 3

$$\begin{array}{r} [0 \quad 2.667 \quad 0.5556 \quad : \quad 53.10] \\ - [0 \quad 2.667 \quad 0.7556 \quad : \quad 53.33] \\ \hline [0 \quad 0 \quad -0.2 \quad : \quad -0.23] \end{array}$$

Substitute new equation for  
Equation 3

$$\left[ \begin{array}{ccc|c} 144 & 12 & 1 & 279.2 \\ 0 & 2.917 & 0.8264 & 58.33 \\ 0 & 0 & -0.2 & -0.23 \end{array} \right]$$

# Back Substitution

# Back Substitution

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for  $a_3$

$$-0.2a_3 = -0.23$$

$$a_3 = \frac{-0.23}{-0.2}$$

$$= 1.15$$

# Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for  $a_2$

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$\begin{aligned} a_2 &= \frac{58.33 - 0.8264a_3}{2.917} \\ &= \frac{58.33 - 0.8264 \times 1.15}{2.917} \\ &= 19.67 \end{aligned}$$

# Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for  $a_1$

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$a_1 = \frac{279.2 - 12a_2 - a_3}{144}$$

$$= \frac{279.2 - 12 \times 19.67 - 1.15}{144}$$

$$= 0.2917$$

# Gaussian Elimination with Partial Pivoting Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

# Gauss Elimination with Partial Pivoting Another Example

<http://numericalmethods.eng.usf.edu>



# Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

# Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

$|10|$ ,  $|-3|$ , and  $|5|$  or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

# Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

$|-0.001|$  and  $|2.5|$  or  $0.0001$  and  $2.5$

The largest absolute value is  $2.5$ , so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

# Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

# Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

# Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{\text{calculated}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad [X]_{\text{exact}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

**THE END**

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# Determinant of a Square Matrix Using Naïve Gauss Elimination Example

<http://numericalmethods.eng.usf.edu>



# Theorem of Determinants

If a multiple of one row of  $[A]_{n \times n}$  is added or subtracted to another row of  $[A]_{n \times n}$  to result in  $[B]_{n \times n}$  then  $\det(A) = \det(B)$

# Theorem of Determinants

The determinant of an upper triangular matrix  $[A]_{n \times n}$  is given by

$$\begin{aligned}\det(A) &= a_{11} \times a_{22} \times \dots \times a_{ii} \times \dots \times a_{nn} \\ &= \prod_{i=1}^n a_{ii}\end{aligned}$$

# Forward Elimination of a Square Matrix

Using forward elimination to transform  $[A]_{n \times n}$  to an upper triangular matrix,  $[U]_{n \times n}$ .

$$[A]_{n \times n} \rightarrow [U]_{n \times n}$$

$$\det(A) = \det(U)$$

# Example

Using naïve Gaussian elimination find the determinant of the following square matrix.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

# Forward Elimination

# Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64,  $\frac{64}{25} = 2.56$ .

$$[25 \quad 5 \quad 1] \times 2.56 = [64 \quad 12.8 \quad 2.56]$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 \end{bmatrix} \\ - \begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

# Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144,  $\frac{144}{25} = 5.76$ .

$$[25 \quad 5 \quad 1] \times 5.76 = [144 \quad 28.8 \quad 5.76]$$

Subtract the result from  
Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 \end{bmatrix} \\ - \begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \end{array}$$

Substitute new equation for  
Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

# Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Divide Equation 2 by  $-4.8$   
and multiply it by  $-16.8$ ,  
 $\frac{-16.8}{-4.8} = 3.5$ .

$$([0 \quad -4.8 \quad -1.56]) \times 3.5 = [0 \quad -16.8 \quad -5.46]$$

Subtract the result from  
Equation 3

$$\begin{array}{r} [0 \quad -16.8 \quad -4.76] \\ - [0 \quad -16.8 \quad -5.46] \\ \hline [0 \quad 0 \quad 0.7] \end{array}$$

Substitute new equation for  
Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$



# Finding the Determinant

After forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= u_{11} \times u_{22} \times u_{33} \\ &= 25 \times (-4.8) \times 0.7 \\ &= -84.00 \end{aligned}$$

# Summary

- Forward Elimination
- Back Substitution
- Pitfalls
- Improvements
- Partial Pivoting
- Determinant of a Matrix

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

[http://numericalmethods.eng.usf.edu/topics/gaussian\\_elimination.html](http://numericalmethods.eng.usf.edu/topics/gaussian_elimination.html)

**THE END**

<http://numericalmethods.eng.usf.edu>