

## Chapter 04.10

# Eigenvalues and Eigenvectors

After reading this chapter, you should be able to:

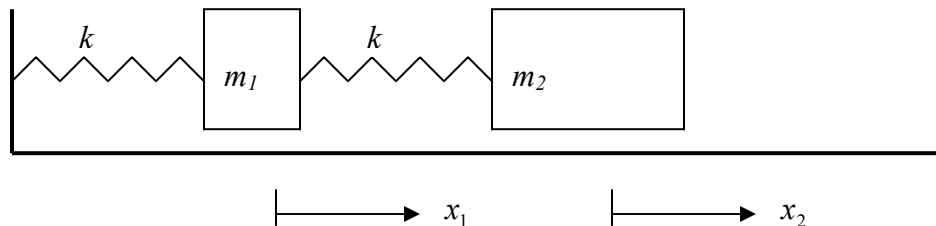
1. define eigenvalues and eigenvectors of a square matrix,
2. find eigenvalues and eigenvectors of a square matrix,
3. relate eigenvalues to the singularity of a square matrix, and
4. use the power method to numerically find the largest eigenvalue in magnitude of a square matrix and the corresponding eigenvector.

### What does eigenvalue mean?

The word eigenvalue comes from the German word *Eigenwert* where Eigen means *characteristic* and Wert means *value*. However, what the word means is not on your mind! You want to know why I need to learn about eigenvalues and eigenvectors. Once I give you an example of an application of eigenvalues and eigenvectors, you will want to know how to find these eigenvalues and eigenvectors.

### Can you give me a physical example application of eigenvalues and eigenvectors?

Look at the spring-mass system as shown in the picture below.



Assume each of the two mass-displacements to be denoted by  $x_1$  and  $x_2$ , and let us assume each spring has the same spring constant  $k$ . Then by applying Newton's 2<sup>nd</sup> and 3<sup>rd</sup> law of motion to develop a force-balance for each mass we have

$$m_1 \frac{d^2 x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1)$$

Rewriting the equations, we have

$$m_1 \frac{d^2 x_1}{dt^2} - k(-2x_1 + x_2) = 0$$

$$m_2 \frac{d^2 x_2}{dt^2} - k(x_1 - x_2) = 0$$

Let  $m_1 = 10, m_2 = 20, k = 15$

$$10 \frac{d^2 x_1}{dt^2} - 15(-2x_1 + x_2) = 0$$

$$20 \frac{d^2 x_2}{dt^2} - 15(x_1 - x_2) = 0$$

From vibration theory, the solutions can be of the form

$$x_i = A_i \sin(\omega t - \theta)$$

where

$A_i$  = amplitude of the vibration of mass  $i$ ,  
 $\omega$  = frequency of vibration,  
 $\theta$  = phase shift.

then

$$\frac{d^2 x_i}{dt^2} = -A_i \omega^2 \sin(\omega t - \theta)$$

Substituting  $x_i$  and  $\frac{d^2 x_i}{dt^2}$  in equations,

$$-10A_1\omega^2 - 15(-2A_1 + A_2) = 0$$

$$-20A_2\omega^2 - 15(A_1 - A_2) = 0$$

gives

$$(-10\omega^2 + 30)A_1 - 15A_2 = 0$$

$$-15A_1 + (-20\omega^2 + 15)A_2 = 0$$

or

$$(-\omega^2 + 3)A_1 - 1.5A_2 = 0$$

$$-0.75A_1 + (-\omega^2 + 0.75)A_2 = 0$$

In matrix form, these equations can be rewritten as

$$\begin{bmatrix} -\omega^2 + 3 & -1.5 \\ -0.75 & -\omega^2 + 0.75 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let  $\omega^2 = \lambda$

$$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

$$[X] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$[A][X] - \lambda[X] = 0$$

$$[A][X] = \lambda[X]$$

In the above equation,  $\lambda$  is the eigenvalue and  $[X]$  is the eigenvector corresponding to  $\lambda$ . As you can see, if we know  $\lambda$  for the above example we can calculate the natural frequency of the vibration

$$\omega = \sqrt{\lambda}$$

Why are the natural frequencies of vibration important? Because you do not want to have a forcing force on the spring-mass system close to this frequency as it would make the amplitude  $A_i$  very large and make the system unstable.

### What is the general definition of eigenvalues and eigenvectors of a square matrix?

If  $[A]$  is a  $n \times n$  matrix, then  $[X] \neq \bar{0}$  is an eigenvector of  $[A]$  if

$$[A][X] = \lambda[X]$$

where  $\lambda$  is a scalar and  $[X] \neq 0$ . The scalar  $\lambda$  is called the eigenvalue of  $[A]$  and  $[X]$  is called the eigenvector corresponding to the eigenvalue  $\lambda$ .

### How do I find eigenvalues of a square matrix?

To find the eigenvalues of a  $n \times n$  matrix  $[A]$ , we have

$$[A][X] = \lambda[X]$$

$$[A][X] - \lambda[X] = 0$$

$$[A][X] - \lambda[I][X] = 0$$

$$([A] - [\lambda][I])[X] = 0$$

Now for the above set of equations to have a nonzero solution,

$$\det([A] - \lambda[I]) = 0$$

This left hand side can be expanded to give a polynomial in  $\lambda$  solving the above equation would give us values of the eigenvalues. The above equation is called the characteristic equation of  $[A]$ .

For a  $[A]$   $n \times n$  matrix, the characteristic polynomial of  $A$  is of degree  $n$  as follows

$$\det([A] - \lambda[I]) = 0$$

giving

$$\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_n = 0$$

Hence, this polynomial has  $n$  roots.

### Example 1

Find the eigenvalues of the physical problem discussed in the beginning of this chapter, that is, find the eigenvalues of the matrix

$$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

**Solution**

$$[A] - \lambda[I] = \begin{bmatrix} 3 - \lambda & -1.5 \\ -0.75 & 0.75 - \lambda \end{bmatrix}$$

$$\det([A] - \lambda[I]) = (3 - \lambda)(0.75 - \lambda) - (-0.75)(-1.5) = 0$$

$$2.25 - 0.75\lambda - 3\lambda + \lambda^2 - 1.125 = 0$$

$$\lambda^2 - 3.75\lambda + 1.125 = 0$$

$$\lambda = \frac{-(-3.75) \pm \sqrt{(-3.75)^2 - 4(1)(1.125)}}{2(1)}$$

$$= \frac{3.75 \pm 3.092}{2}$$

$$= 3.421, 0.3288$$

So the eigenvalues are 3.421 and 0.3288.

**Example 2**

Find the eigenvectors of

$$A = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

**Solution**

The eigenvalues have already been found in [Example 10.1](#) as

$$\lambda_1 = 3.421, \lambda_2 = 0.3288$$

Let

$$[X] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the eigenvector corresponding to

$$\lambda_1 = 3.421$$

Hence

$$([A] - \lambda_1[I])[X] = 0$$

$$\left\{ \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix} - 3.421 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.421 & -1.5 \\ -0.75 & -2.671 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If

$$x_1 = s$$

then

$$-0.421s - 1.5x_2 = 0$$

$$x_2 = -0.2808s$$

The eigenvector corresponding to  $\lambda_1 = 3.421$  then is

$$\begin{aligned}
 [X] &= \begin{bmatrix} s \\ -0.2808s \end{bmatrix} \\
 &= s \begin{bmatrix} 1 \\ -0.2808 \end{bmatrix}
 \end{aligned}$$

The eigenvector corresponding to  
 $\lambda_1 = 3.421$

is

$$\begin{bmatrix} 1 \\ -0.2808 \end{bmatrix}$$

Similarly, the eigenvector corresponding to  
 $\lambda_2 = 0.3288$

is

$$\begin{bmatrix} 1 \\ 1.781 \end{bmatrix}$$

### Example 3

Find the eigenvalues and eigenvectors of

$$[A] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}$$

#### Solution

The characteristic equation is given by

$$\det([A] - \lambda[I]) = 0$$

$$\det \begin{bmatrix} 1.5 - \lambda & 0 & 1 \\ -0.5 & 0.5 - \lambda & -0.5 \\ -0.5 & 0 & -\lambda \end{bmatrix} = 0$$

$$(1.5 - \lambda)[(0.5 - \lambda)(-\lambda) - (-0.5)(0)] + (1)[(-0.5)(0) - (-0.5)(0.5 - \lambda)] = 0$$

$$-\lambda^3 + 2\lambda^2 - 1.25\lambda = 0$$

The roots of the above equation are

$$\lambda = 0.5, 0.5, 1.0$$

Note that there are eigenvalues that are repeated. Since there are only two distinct eigenvalues, there are only two eigenspaces. But, corresponding to  $\lambda = 0.5$  there should be two eigenvectors that form a basis for the eigenspace.

To find the eigenspaces, let

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Given

$$[(A - \lambda I)][X] = 0$$

then

$$\begin{bmatrix} 1.5 - \lambda & 0 & 1 \\ -0.5 & 0.5 - \lambda & -0.5 \\ -0.5 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For  $\lambda = 0.5$ ,

$$\begin{bmatrix} 1 & 0 & 1 \\ -0.5 & 0 & -0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives

$$x_1 = -a, x_2 = b, x_3 = a$$

So

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -a \\ b \\ a \end{bmatrix} \\ &= \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \\ &= a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

So the vectors  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  form a basis for the eigenspace for the eigenvalue  $\lambda = 0.5$ .

For  $\lambda = 1$ ,

$$\begin{bmatrix} 0.5 & 0 & 1 \\ -0.5 & -0.5 & -0.5 \\ -0.5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives

$$x_1 = a, x_2 = -0.5a, x_3 = -0.5a$$

The eigenvector corresponding to  $\lambda = 1$  is

$$\begin{bmatrix} a \\ -0.5a \\ -0.5a \end{bmatrix} = a \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

Hence the vector  $\begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$  is a basis for the eigenspace for the eigenvalue of  $\lambda = 1$ .

**What are some of the theorems of eigenvalues and eigenvectors?**

Theorem 1: If  $[A]$  is a  $n \times n$  triangular matrix – upper triangular, lower triangular or diagonal, the eigenvalues of  $[A]$  are the diagonal entries of  $[A]$ .

Theorem 2:  $\lambda = 0$  is an eigenvalue of  $[A]$  if  $[A]$  is a singular (noninvertible) matrix.

Theorem 3:  $[A]$  and  $[A]^T$  have the same eigenvalues.

Theorem 4: Eigenvalues of a symmetric matrix are real.

Theorem 5: Eigenvectors of a symmetric matrix are orthogonal, but only for distinct eigenvalues.

Theorem 6:  $|\det(A)|$  is the product of the absolute values of the eigenvalues of  $[A]$ .

**Example 4**

What are the eigenvalues of

$$[A] = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 5 & 7.5 & 0 \\ 2 & 6 & 0 & -7.2 \end{bmatrix}$$

**Solution**

Since the matrix  $[A]$  is a lower triangular matrix, the eigenvalues of  $[A]$  are the diagonal elements of  $[A]$ . The eigenvalues are

$$\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 7.5, \lambda_4 = -7.2$$

**Example 5**

One of the eigenvalues of

$$[A] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 5 & 9 \\ 2 & 1 & -7 \end{bmatrix}$$

is zero. Is  $[A]$  invertible?

**Solution**

$\lambda = 0$  is an eigenvalue of  $[A]$ , that implies  $[A]$  is singular and is not invertible.

**Example 6**

Given the eigenvalues of

$$[A] = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$$

are

$$\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$$

What are the eigenvalues of  $[B]$  if

$$[B] = \begin{bmatrix} 2 & 3.5 & 8 \\ -3.5 & 5 & 1 \\ 6 & 2 & 8.5 \end{bmatrix}$$

**Solution**

Since  $[B] = [A]^T$ , the eigenvalues of  $[A]$  and  $[B]$  are the same. Hence eigenvalues of  $[B]$  also are

$$\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$$

**Example 7**

Given the eigenvalues of

$$[A] = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$$

are

$$\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$$

Calculate the magnitude of the determinant of the matrix.

**Solution**

Since

$$\begin{aligned} |\det[A]| &= |\lambda_1| |\lambda_2| |\lambda_3| \\ &= |-1.547| |12.33| |4.711| \\ &= 89.88 \end{aligned}$$

**How does one find eigenvalues and eigenvectors numerically?**

One of the most common methods used for finding eigenvalues and eigenvectors is the power method. It is used to find the largest eigenvalue in an absolute sense. \*Note that if this largest eigenvalue is repeated, this method will not work. Also this eigenvalue needs to be distinct. The method is as follows:

1. Assume a guess  $[X^{(0)}]$  for the eigenvector in  
 $[A][X] = \lambda[X]$   
 equation. One of the entries of  $[X^{(0)}]$  needs to be unity.
2. Find  
 $[Y^{(1)}] = [A][X^{(0)}]$
3. Scale  $[Y^{(1)}]$  so that the chosen unity component remains unity.  
 $[Y^{(1)}] = \lambda^{(1)}[X^{(1)}]$
4. Repeat steps (2) and (3) with  
 $[X] = [X^{(1)}]$  to get  $[X^{(2)}]$ .
5. Repeat the steps 2 and 3 until the value of the eigenvalue converges.



If  $E_s$  is the pre-specified percentage relative error tolerance to which you would like the answer to converge to, keep iterating until

$$\left| \frac{\lambda^{(i+1)} - \lambda^{(i)}}{\lambda^{(i+1)}} \right| \times 100 \leq E_s$$

where the left hand side of the above inequality is the definition of absolute percentage relative approximate error, denoted generally by  $E_s$ . A pre-specified percentage relative tolerance of  $0.5 \times 10^{2-m}$  implies at least  $m$  significant digits are current in your answer. When the system converges, the value of  $\lambda$  is the largest (in absolute value) eigenvalue of  $[A]$ .

### Example 8

Using the power method, find the largest eigenvalue and the corresponding eigenvector of

$$[A] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}$$

### Solution

Assume

$$[X^{(0)}] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[A][X^{(0)}] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$Y^{(1)} = 2.5 \begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$$

$$\lambda^{(1)} = 2.5$$

We will choose the first element of  $[X^{(0)}]$  to be unity.

$$[X^{(1)}] = \begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$$

$$\begin{aligned}
 [A][X^{(1)}] &= \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix} \\
 &= \begin{bmatrix} 1.3 \\ -0.5 \\ -0.5 \end{bmatrix} \\
 [X^{(2)}] &= 1.3 \begin{bmatrix} 1 \\ -0.3846 \\ -0.3846 \end{bmatrix}
 \end{aligned}$$

$$\lambda^{(2)} = 1.3$$

$$[X^{(2)}] = \begin{bmatrix} 1 \\ -0.3846 \\ -0.3846 \end{bmatrix}$$

The absolute relative approximate error in the eigenvalues is

$$\begin{aligned}
 |\varepsilon_a| &= \left| \frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}} \right| \times 100 \\
 &= \left| \frac{1.3 - 1.5}{1.5} \right| \times 100 \\
 &= 92.307\%
 \end{aligned}$$

Conducting further iterations, the values of  $\lambda^{(i)}$  and the corresponding eigenvectors is given in the table below

$i$	$\lambda^{(i)}$	$[X^{(i)}]$	$ \varepsilon_a $ (%)
1	2.5	$\begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$	—
2	1.3	$\begin{bmatrix} 1 \\ -0.38462 \\ -0.38462 \end{bmatrix}$	92.307
3	1.1154	$\begin{bmatrix} 1 \\ -0.44827 \\ -0.44827 \end{bmatrix}$	16.552
4	1.0517	$\begin{bmatrix} 1 \\ -0.47541 \\ -0.47541 \end{bmatrix}$	6.0529

5	1.02459	$\begin{bmatrix} 1 \\ -0.48800 \\ -0.48800 \end{bmatrix}$	1.2441
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The exact value of the eigenvalue is  $\lambda = 1$   
and the corresponding eigenvector is

$$[X] = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

**Key Terms:**

*Eigenvalue*

*Eigenvectors*

*Power method*