Chapter 09.00C

Physical Problem for Optimization
Civil Engineering

Problem Statement

In civil engineering applications, the channel flow is an important problem for irrigation water supply and flood drainage systems. An engineer seeks to minimize the wetted perimeter of a channel to create the least resistance to the flow, but to increase the volume flow rate, one needs to increase the cross-sectional flow area. Consider a trapezoidal channel as shown in Figure 1. A channel cross-sectional shape is shown in Figure 1, and one wishes to minimize the wetted perimeter. What are the values of s and θ, that will minimize the wetted perimeter for a fixed cross-section of $4\,m^2$?

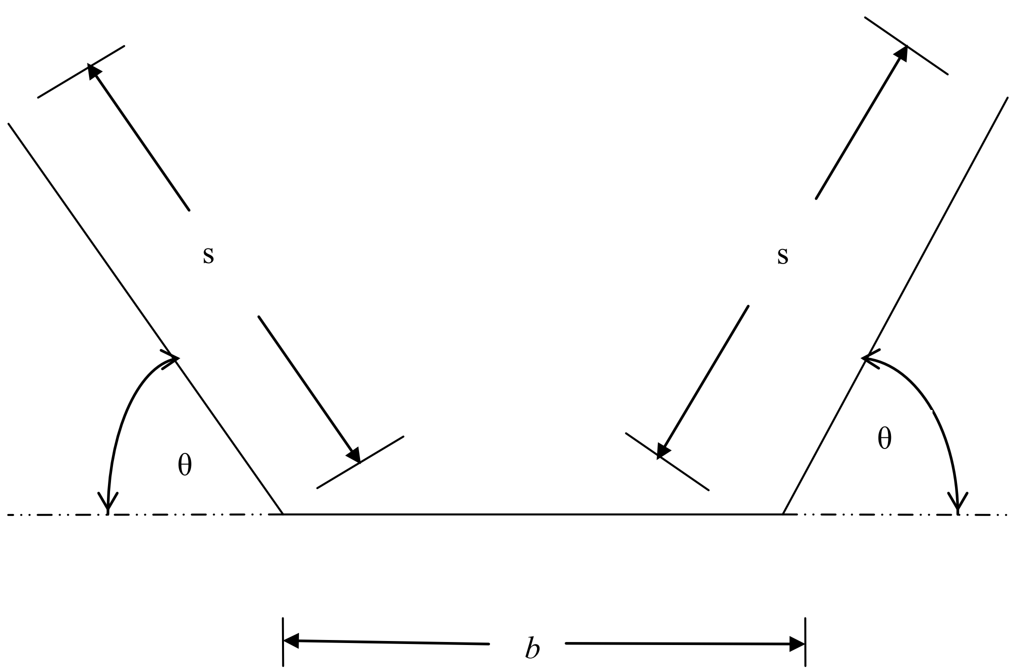

Figure 1: A trapezoidal cross-section of the channel

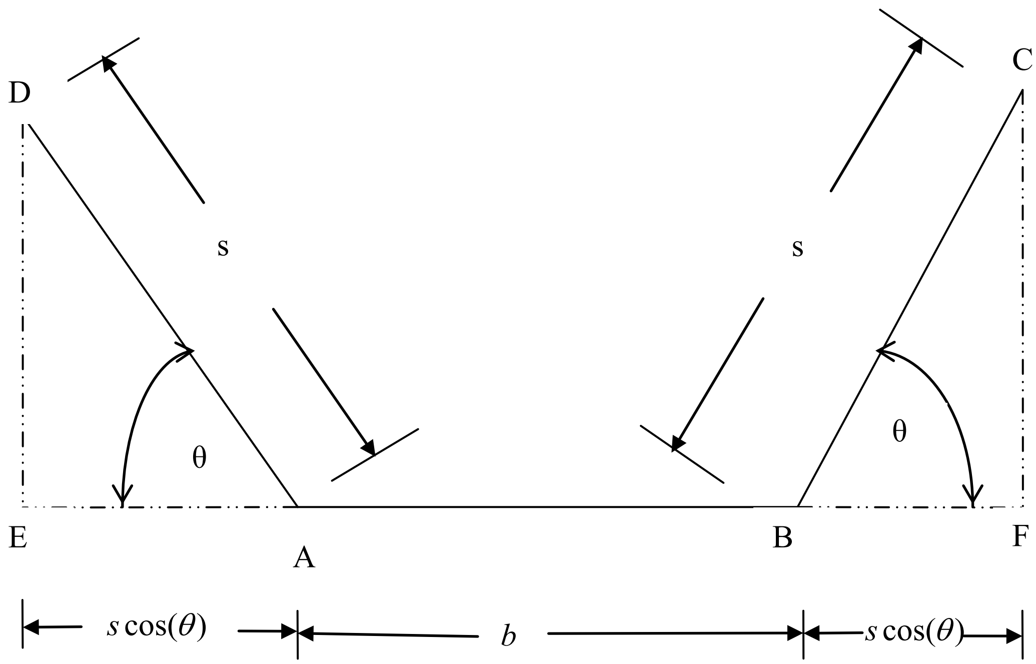

Figure 2: Labeling of the trapezoidal water channel

Solution

The cross-sectional area of the trapezoid is the cross-sectional area G of the channel, given by

$$G = \text{Area of trapezoid } ABCD$$
$$= \frac{1}{2}(AB + CD)(FC)$$

where

$$AB = b$$
$$CD = AB + EA + BF$$
$$= b + s\cos(\theta) + s\cos(\theta)$$
$$= b + 2s\cos(\theta)$$
$$FC = BC \sin(\theta)$$
$$= s\sin(\theta)$$

Hence

$$G = \frac{1}{2}(AB + CD)\ (DE)$$

$$G(s,\theta) = \frac{1}{2}\left[b + b + 2s\cos(\theta)\right] s \sin(\theta)$$

$$= [b + s\cos(\theta)]s\sin(\theta) \quad (1)$$

The length of the wetted perimeter is given by

$$L = b + 2s \quad (2)$$

The optimization problem now is to minimize L while constraining the cross-sectional area. Let us assume the cross-sectional area is set at $4\,m^2$. Then from Equation (1)

$$[b + s\cos(\theta)]s\sin(\theta) = 4$$

giving

$$b = \frac{4}{s\sin(\theta)} - s\cos(\theta) \quad (3)$$

substituting the value of b from Equation (3) into Equation (2) gives

$$L(s,\theta) = \frac{4}{s\sin(\theta)} - s\cos(\theta) + 2s \quad (4)$$

Hence we now have an unconstrained optimization problem of finding s and θ, such that L is minimized.

Equation (4) models a two-dimensional optimization problem where we want to minimize L, and the value of s and θ can be chosen to do so. Since we are going to cover both one-dimensional and two-dimensional optimization problems, a one-dimensional version would be where we fix one of the two variables. One case would be where we choose

$$b = s, \quad (5)$$

Then from Equation (1)

$$[b + b\cos(\theta)]b\sin(\theta) = 4$$

giving

$$b = \sqrt{\frac{4}{(1+\cos(\theta))\sin(\theta)}} \quad (6)$$

Substituting the value of b from Equation (6) in Equation (2), and noting the equality in Equation (4) gives

$$\begin{aligned} L(s,\theta) &= b + 2s \\ &= b + 2b \\ &= 3b \quad (7) \\ &= \sqrt{\frac{12}{(1+\cos(\theta))\sin(\theta)}} \end{aligned}$$

Questions

1) Find the optimal value of θ, such that the wetted perimeter of the channel is minimized. Use the expression given in Equation (6).

2) Find the optimum values of s and θ, such that the wetted perimeter of the channel is minimized. Use the expression given in Equation (7).

OPTIMIZATION	
Topic	Physical problem
Summary	A physical problem to find the minimum wetted perimeter.
Major	Civil Engineering
Authors	Autar Kaw
Date	December 5, 2011
Web Site	http://numericalmethods.eng.usf.edu