Chapter 11.00C

Physical Problem for Fast Fourier Transform Civil Engineering

Introduction

In this chapter, applications of FFT algorithms [1-5] for solving real-life problems such as computing the dynamical (displacement) response [6-7] of single degree of freedom (SDOF) water tower structure will be demonstrated.

Free Vibration Response of Single Degree-Of- Freedom, (SDOF) Systems

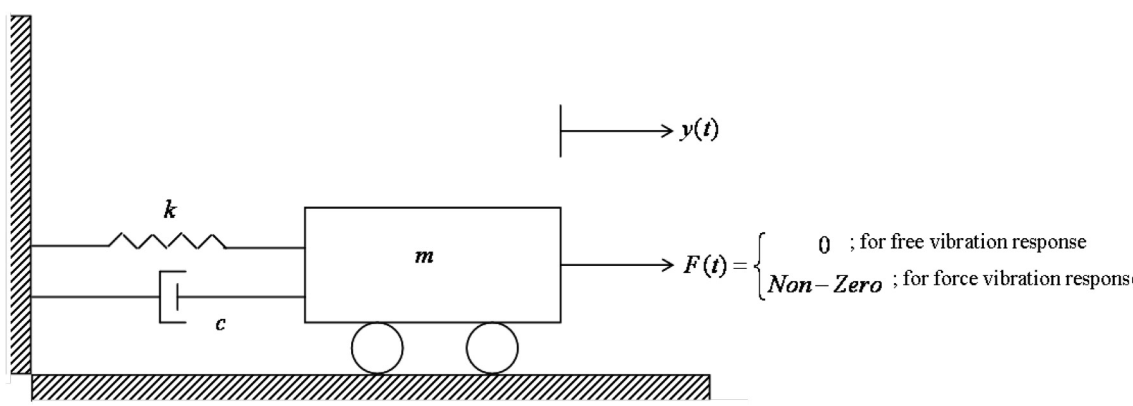

Figure 1 SDOF dynamic (water tower structure) system.

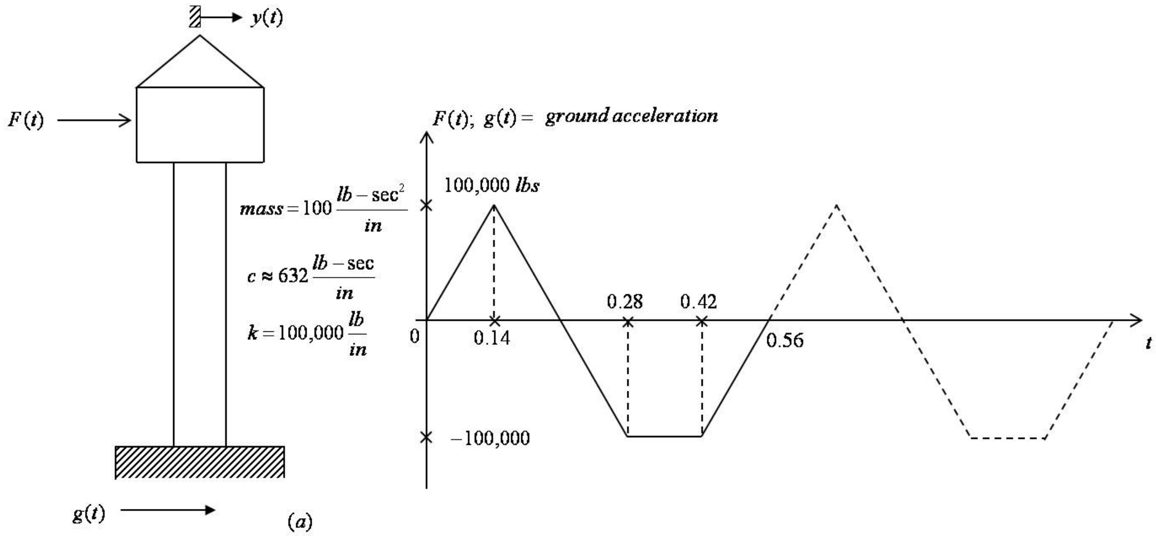

Figure 2 Water tower structure subjected to dynamic loads.

a) Water tower structure, Idealized as SDOF system.
b) Impulse blast loading $F(t)$, or earthquake ground acceleration $g(t)$.

The dynamical equilibrium for a SDOF system (shown in Figure 1) can be given as:

$$m\ddot{y} + c\dot{y} + ky = F(t) = F_0 \sin(\overline{w}t) \tag{1}$$

where

m, c and $k =$ mass, damping and spring stiffness, respectively (which are related to inertia, damping and spring forces, respectively).

$y, \dot{y}, \ddot{y} =$ displacement, velocity, and acceleration, respectively.

Practical structural models such as the water tower structure subjected to applied blast loading (or earthquake ground acceleration) etc. can be conveniently modeled and studied as a simple SDOF system (shown in Figure 2).

For free vibration response, Equation (1) simplifies to

$$\begin{aligned} m\ddot{y} + c\dot{y} + ky &= F(t) \\ &= 0 \end{aligned} \tag{2}$$

The solution (displacement response y) of Equation (2) can be expressed as

$$y(t) = Qe^{pt} = displacement \tag{3}$$

Hence

$$\dot{y} = Qpe^{pt} = velocity = \frac{dy}{dt} \tag{4}$$

$$\ddot{y} = Qp^2e^{pt} = acceleration = \frac{d^2y}{dt^2} \tag{5}$$

Substituting Equations (3-5) into Equation (2), one obtains

$$mp^2 + cp + k = 0 \tag{6}$$

The two roots of the above quadratic equation can be obtained as

$$p = \frac{-c \pm \sqrt{c^2 - 4(m)(k)}}{2m} \tag{7}$$

$$= \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{8}$$

Critical Damping (C_{cr})

In this case, the term under the square root in Equation (8) is set to be zero, hence

$$\left(\frac{C_{cr}}{2m}\right)^2 - \frac{k}{m} = 0 \tag{9}$$

or

$$C_{cr} = 2\sqrt{km} \tag{10}$$

since

$$w = \sqrt{\frac{k}{m}} \tag{11}$$

Hence

$$C_{cr} = 2mw \tag{12}$$

$$= \frac{2k}{w}$$

The two identical roots of Equation (8) can be computed as

$$p_1, p_2 = \frac{-C_{cr}}{2m} \tag{13}$$

and the solution $y(t)$ in Equation (3) can be given as

$$y(t) = Q_1 e^{p_1 t} + Q_2 t e^{p_2 t} \tag{14}$$

$$= (Q_1 + Q_2 t) e^{\left(-\frac{C_{cr}}{2m}\right)t} \tag{15}$$

which can be plotted as shown in Figure 3.

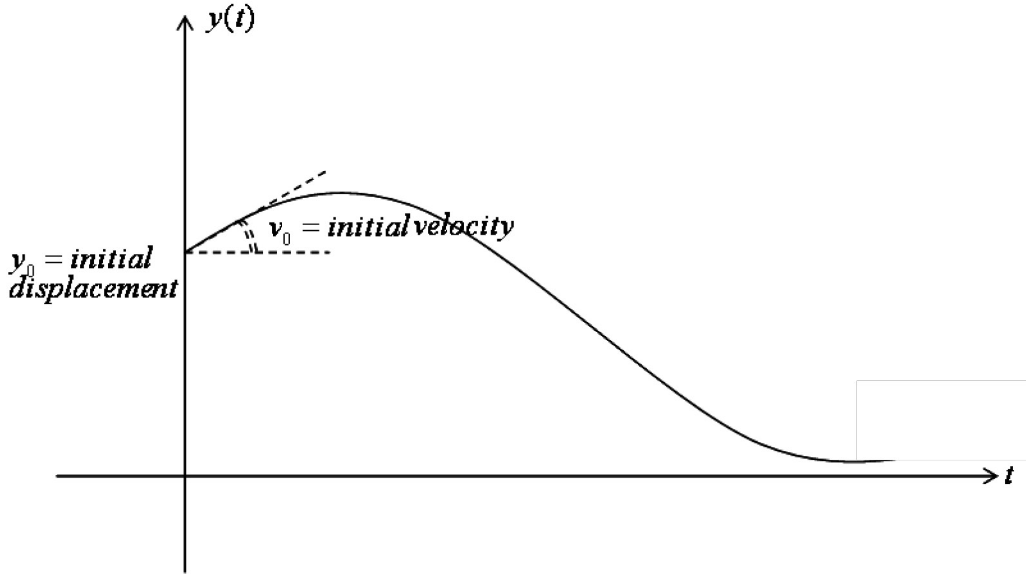

Figure 3 Free vibration with critical damping.

Over damping $\left(C > C_{cr}\right)$

In this case, one has

$$\left(\frac{C}{2m}\right)^2 - \frac{k}{m} > 0 \tag{16}$$

The solution of $y(t)$ from Equation (3) can be given as

$$y(t) = Q_1 e^{p_1 t} + Q_2 e^{p_2 t} \tag{17}$$

The response of over damping system is similar to Figure 3.

Under Damping $\left(C < C_{cr}\right)$

In this case, one has

$$\left(\frac{C}{2m}\right)^2 - \frac{k}{m} < 0 \tag{18}$$

and the two "complex" roots from Equation (8) can be given as

$$p_1, p_2 = -\frac{C}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{C}{2m}\right)^2} \tag{19}$$

Substituting Equation (19), and using Euler's equation $\left[e^{i\theta} = \cos(\theta) + i\sin(\theta)\right]$, Equation (3) or Equation (17) becomes

$$y(t) = e^{-(c/2m)t}\left(A\cos w_D t + B\sin w_D t\right) \tag{20}$$

where

$$w_D = \sqrt{\frac{k}{m} - \left(\frac{C}{2m}\right)^2} \quad \text{see Equation (19)} \tag{21}$$

$$= w\sqrt{1-\xi^2} \qquad (22)$$

$$\xi = \frac{C}{C_{cr}}$$

$$= \frac{C}{2\sqrt{km}} \qquad (23)$$

Using the initial conditions:

$$@t = 0; y = y_0; \dot{y} = v_0 \qquad (24)$$

Then, the two constants (A and B) can be solved, and Equation (20) becomes

$$y(t) = e^{-\xi wt}\left(y_0 \cos w_D t + \frac{v_0 + y_0 \xi w}{w_D} \times \sin w_D t\right) \qquad (25)$$

Equation (11.216) can also be expressed as:

$$y(t) = K_1 e^{-\xi wt} \times \cos(w_D t - \alpha) \qquad (26)$$

where

$$K_1 = \sqrt{y_0^2 + \frac{(v_0 + y_0 \xi w)^2}{w_D^2}} \qquad (27)$$

$$\tan(\alpha) = \frac{v_0 + y_0 \xi w}{w_D y_0} \qquad (28)$$

Equation (26) can be plotted as shown in Figure 4.

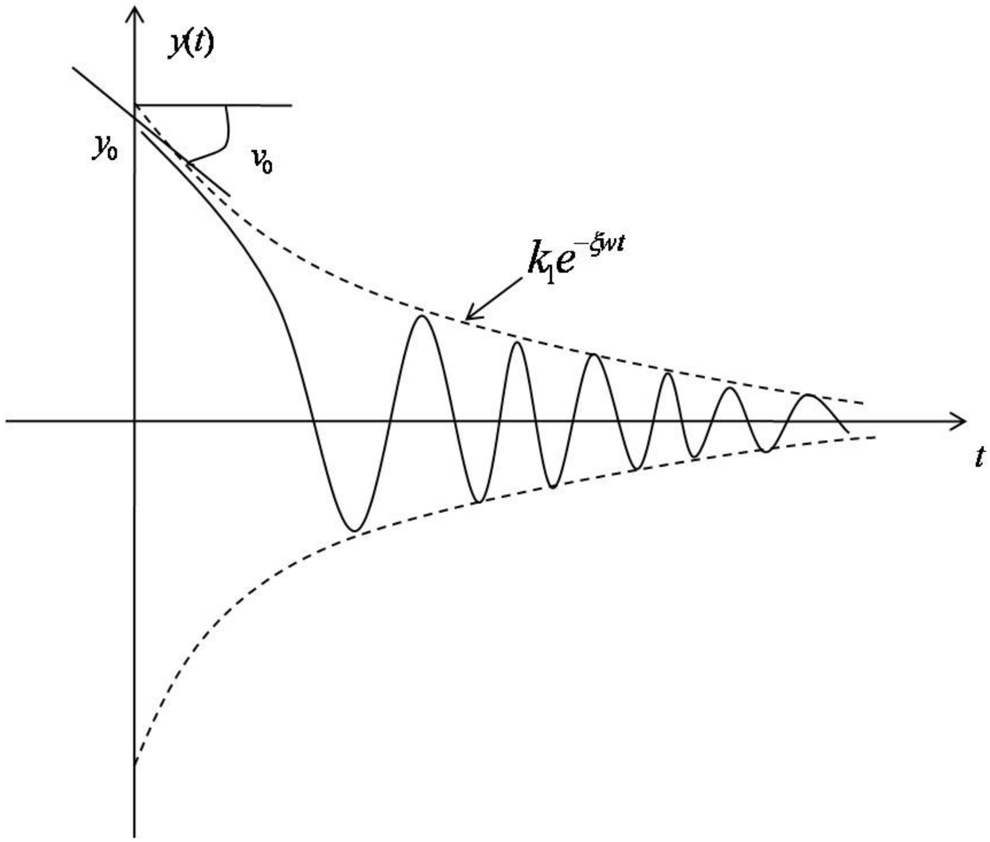

Figure 4 Free vibration of SDOF under damped system.

Force Vibration Response of SDOF Systems

For force vibration problem, the right-hand-side (RHS) of Equation (1) $F(t) \neq 0,$ and the general solution for Equation (1) can be given as

$$y(t) = y_c(t) + y_p(t) \tag{29}$$

where the complimentary solution $y_c(t)$ can be obtained as (see. Equation (20)) assumed under-damped $(C < C_{cr})$ case

$$y_c(t) = e^{-(C/2m)t}(A\cos w_D t + B\sin w_D t) \qquad \text{(20, repeated)}$$

$$= e^{\left(\frac{-C\sqrt{k}}{\sqrt{k}} * \frac{t}{2\sqrt{m}\sqrt{m}}\right)}(A\cos w_D t + B\sin w_D t)$$

$$= e^{-C(\sqrt{k/m})t/2\sqrt{km}}(A\cos w_D t + B\sin w_D t) \tag{30}$$

Using Equations (10) and (11), Equation (30) becomes

$$y_c(t) = e^{-Cwt/C_{cr}}(A\cos w_D t + B\sin w_D t)$$

Using Equation (23), the above equation becomes

$$y_c(t) = e^{-\xi wt}(A\cos w_D t + B\sin w_D t) \tag{31}$$

The particular solution $y_p(t)$, associated with the particular sine term forcing function $F(t) = F_0 \sin(\overline{w}t)$ see Equation (1) can be given as

$$y_p(t) = C_1 \sin(\overline{w}t) + C_2 \cos(\overline{w}t) \tag{32}$$

The unknown constants C_1 and C_2 can be found by substituting Equation (32) into Equation (1), and equating the coefficients of the sine and cosine functions.

Using Euler's identity, one has

$$e^{i\overline{w}t} = \cos(\overline{w}t) + i\sin(\overline{w}t) \tag{33}$$

Thus, the RHS of Equation (1) can be expressed as

$$m\ddot{y} + c\dot{y} + ky = F_0 \times \text{ Imaginary portion of } e^{i\overline{w}t} \tag{34}$$

Hence, the response will consist of ONLY the imaginary portion of Equation (29).
The particular solution $y_p(t)$, shown in Equation (32), can be more conveniently expressed as

$$y_p(t) = C^* e^{i\overline{w}t} \tag{35}$$

Substituting Equation (35) into Equation (34), one gets

$$m\left\{C^* i^2 \overline{w}^2 e^{i\overline{w}t}\right\} + c\left\{C^* i\overline{w} e^{i\overline{w}t}\right\} + k\left\{C^* e^{i\overline{w}t}\right\} = F_0 \times e^{i\overline{w}t} \tag{36}$$

or

$$C^*\left\{k + ic\overline{w} - m\overline{w}^2\right\} = F_0 \tag{37}$$

Hence

$$C^* = \frac{F_0}{k - m\overline{w}^2 + ic\overline{w}} \tag{38}$$

Substituting Equation (38) into Equation (35), one obtains

$$y_p(t) = \left(\frac{F_0}{k - m\overline{w}^2 + ic\overline{w}}\right) e^{i\overline{w}t} \tag{39}$$

In Equation (39), the "complex" number

$$d \equiv \left(k - m\overline{w}^2\right) + i\left(c\overline{w}\right) \tag{40}$$

can be symbolically expressed as

$$d = (d_R) + i(d_I) \tag{41}$$

or in polar coordinates, one has (see Figure 5)

$$d = |d| e^{i\theta} = |d| \times \{\cos(\theta) + i\sin(\theta)\} \tag{42}$$

$$\tan(\theta) \equiv \frac{\sin(\theta)}{\cos(\theta)} \tag{43}$$

$$= \frac{c\overline{w}}{k - m\overline{w}^2}$$

where

$$d_R = k - m\overline{w}^2 \tag{44}$$

$$d_I = c\overline{w} \tag{45}$$

$$|d| = \sqrt{(d_R)^2 + (d_I)^2} \tag{46}$$

$$= \sqrt{\left(k - m\overline{w}^2\right)^2 + (c\overline{w})^2} \tag{47}$$

Thus, Equation (39) can be re-written as:

$$y_p(t) = \frac{F_0 e^{-i\overline{w}t}}{\sqrt{\left(k - m\overline{w}^2\right)^2 + (c\overline{w})^2} \times e^{i\theta}} \tag{48}$$

$$= \frac{F_0 e^{i(\overline{w}t - \theta)}}{\sqrt{\left(k - m\overline{w}^2\right)^2 + (c\overline{w})^2}} \tag{49}$$

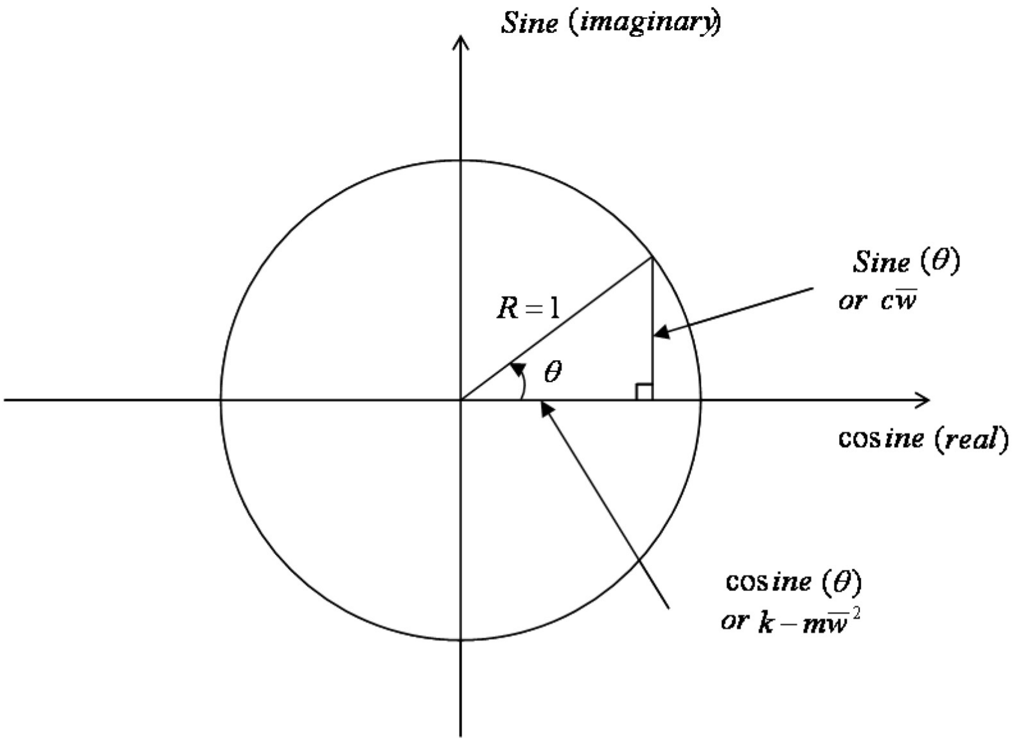

Figure 5 Polar coordinates.

The "imaginary" portion of Equation (49) can be given as

$$y_p(t) = \frac{F_0 \sin(\overline{w}t - \theta)}{\sqrt{\left(k - m\overline{w}^2\right)^2 + (c\overline{w})^2}} \tag{50}$$

Define

$$Y = \frac{F_0}{\sqrt{\left(k - m\overline{w}^2\right)^2 + (c\overline{w})^2}} = \text{amplitude of the steady state motion} \tag{51}$$

$$y_{st} = \frac{F_0}{k} = \text{static deflection of a spring acted by the force } F_0 \tag{52}$$

$$r = \frac{\overline{w}}{w} = \text{frequency ratio (of applied load/structure)} \tag{53}$$

Then, Equations (43) and (50) become

$$\tan(\theta) = \frac{2\xi r}{1-r^2}; \text{ also refer to Equation (23)} \quad (54)$$

$$y_p(t) = Y\sin(\overline{w}t - \theta) \quad (55)$$

$$= \frac{y_{st}\sin(\overline{w}t-\theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad (56)$$

The complimentary (or transient) solution $y_c(t)$ shown in Equation (31), and the particular solution $y_p(t)$ shown in Equation (56) can be substituted into the general solution (see Equation (29)) to obtain

$$y(t) = e^{-\xi wt}(A\cos w_D t + B\sin w_D t) + \frac{y_{st}\sin(\overline{w}t-\theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \quad (57)$$

Define

$$D \equiv \frac{Y}{y_{st}} \quad (58)$$

$$= \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$$

$$D = \text{Dynamic Magnification Factor} \quad (59)$$

Dynamical Response by Fourier Series, DFT and FFT.

The dynamic load $F(t)$ acting on the SDOF system can also be expressed in Fourier series as

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\overline{w}t) + b_n\sin(n\overline{w}t) \quad (60)$$

where the unknown Fourier coefficients can be computed as

$$\left.\begin{aligned} a_0 &= \left(\frac{1}{T}\right)\int_{t_0}^{t_0+T} F(t)dt \\ a_n &= \left(\frac{2}{T}\right)\int_{t_0}^{t_0+T} F(t)\cos(n\overline{w}t)dt \\ b_n &= \left(\frac{2}{T}\right)\int F(t)\sin(n\overline{w}t)dt \end{aligned}\right\} \quad (61)$$

If the forcing function contains only sine terms, then the particular (steady state) solution can be found as (see Equation (56)):

$$y_n = y_{p_n} \quad (62)$$

$$= \left(\frac{b_n}{k}\right)\frac{\sin(n\overline{w}t-\theta)}{\sqrt{\left(1-r_n^2\right)^2 + \left(2r_n\xi\right)^2}}$$

$$=\left(\frac{b_n}{k}\right)\frac{\sin(n\overline{w}t)\cos(\theta)-\sin(\theta)\cos(n\overline{w}t)}{\sqrt{\left(1-r_n^2\right)^2+(2r_n\xi)^2}} \tag{63}$$

Recalled Equation (54), one has

$$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$$
$$=\frac{2\xi r_n}{1-r_n^2}$$

Hence

$$\frac{\sin^2(\theta)}{\cos^2(\theta)}=\frac{\left[\sin^2(\theta)\equiv x^2\right]}{1-\sin^2(\theta)} \tag{64}$$
$$=\frac{1-\cos^2(\theta)}{\left[\cos^2(\theta)\equiv y^2\right]}$$
$$=\frac{\left(2\xi r_n\right)^2}{\left(1-r_n^2\right)^2}$$

Solving Equation (64) for $(x=\sin(\theta))$ and $(y=\cos(\theta))$, one gets

$$x\equiv\sin\theta$$
$$=\frac{2\xi r_n}{\sqrt{\left(1-r_n^2\right)^2+\left(2\xi r_n\right)^2}}$$
$$y\equiv\cos\theta \tag{65}$$
$$=\frac{1-r_n^2}{\sqrt{\left(1-r_n^2\right)^2+\left(2\xi r_n\right)^2}}$$

Substituting Equation (65) into Equation (63) to obtain:

$$y_n(t)=y_{p_n} \tag{66}$$
$$=\left(\frac{b_n}{k}\right)\frac{\left(1-r_n^2\right)\sin\left(n\overline{w}t\right)-\left(2\xi r_n\right)\cos\left(n\overline{w}t\right)}{\left(1-r_n^2\right)^2+\left(2\xi r_n\right)^2}$$

Similarly, if the forcing function contains only the cosine terms, then the particular (steady state) solution can be found as:

$$y_n(t)=y_{p_n} \tag{67}$$
$$=\left(\frac{a_n}{k}\right)\frac{\left(1-r_n^2\right)\cos\left(n\overline{w}t\right)+\left(2\xi r_n\right)\sin\left(n\overline{w}t\right)}{\left(1-r_n^2\right)^2+\left(2\xi r_n\right)^2}$$

Finally, if the forcing function contains both sine and cosine terms, then the total response can be computed by combining both equations (66) and (67), including the constant forcing term a_0, as following

$$y(t) = \sum y_n(t) = \left(\frac{a_0}{k}\right)$$

$$+\left(\frac{1}{k}\right)\sum_{1}^{\infty}\left\{\frac{b_n\left(1-r_n^2\right)+a_n\left(2\xi r_n\right)}{\left(1-r_n^2\right)^2+\left(2\xi r_n\right)^2}\times\sin(n\overline{w}t)+\frac{a_n\left(1-r_n^2\right)-b_n\left(2\xi r_n\right)}{\left(1-r_n^2\right)^2+\left(2\xi r_n\right)^2}\times\cos(n\overline{w}t)\right\} \quad (68)$$

Remarks

Using Euler's relationships, the dynamic load $F(t)$ as shown in Equation (60), can also be expressed in exponential form as

$$F(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\overline{w}t} \quad (18, \text{Ch. } 11.02)$$

where

$$C_n = \left(\frac{1}{T}\right)\int_0^T F(t)e^{-in\overline{w}t}dt \quad (20, \text{Ch. } 11.02)$$

For DFT, define

$$\Delta t = \frac{T}{N} \text{ ; with } t_0, t_1, t_2, \ldots\ldots\ldots t_{N-1} \quad (69)$$

where

$$t_j = j\Delta t \quad (70)$$

Then, the DFT pairs of Equations (21, 1, Ch. 11.04) becomes:

$$\widetilde{C}_k = \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} F(t_n)e^{-ik\left(w_0=\frac{2\pi}{T}\right)t_n}$$

$$= \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} F(t_n)e^{-ik\left(\frac{2\pi}{N\Delta t}\right)n\Delta t}$$

$$= \left(\frac{1}{N}\right)\sum_{j=0}^{N-1} F(t_j)e^{-in\left(\frac{2\pi}{N}\right)j} \text{ ; with } n = 0,1,2,\ldots\ldots N-1 \quad (71)$$

and

$$F(t_j) = \sum_{n=0}^{N-1} \widetilde{C}_n e^{in\left(\frac{2\pi}{N}\right)j} \text{ ; with } j = 0,1,2,\ldots\ldots N-1 \quad (72)$$

Since both Equations 71 and 72 do have similar operations, with the exceptions of the factor $\left(\frac{1}{N}\right)$ and the sign $(-\text{ or }+)$ of the exponential term, both these equations can be handled by the same "general_dft" program given at
http://numericalmethods.eng.usf.edu/simulations/mtl/11fft/fft_civil_engg_example12.m

Introduce the unit amplitude exponential forcing function

$$F(t) = (F_0 = 1) \times e^{iw_n t} \quad (73)$$

into RHS of Equation (1), the steady state solution can also be obtained as (see Equation 39):

$$y(t) = y_p(t) = \left(\frac{1}{k - m\overline{w}_n^2 + ic\overline{w}_n} \right) e^{i\overline{w}_n t} \tag{39, repeated}$$

Using the notations defined in Equations (23) and (53), the above equation can be written as, for a harmonic force component of amplitude $\widetilde{C}_n$.

$$\begin{aligned} y_n(t_j) &= \left\{ \frac{\widetilde{C}_n}{k\left(1 - r_n^2 + i2\xi r_n\right)} \right\} \times e^{i(\overline{w}_n = n\overline{w})(t_j = j\Delta t)} \\ &= \left\{ \frac{\widetilde{C}_n}{k\left(1 - r_n^2 + i2\xi r_n\right)} \right\} \times e^{in\left(\overline{w} = \frac{2\pi}{T}\right) j\Delta t} \\ &= \left\{ \frac{\widetilde{C}_n}{k\left(1 - r_n^2 + i2\xi r_n\right)} \right\} \times e^{in\left(\frac{2\pi}{N\Delta t}\right) j\Delta t} \\ &= \left\{ \frac{\widetilde{C}_n}{k\left(1 - r_n^2 + i2\xi r_n\right)} \right\} \times e^{inj2\pi/N} \end{aligned} \tag{74}$$

and the total (steady state) response due to "n" harmonic force components can be calculated as

$$y(t_j) = \sum_{n=0}^{N-1} \frac{\widetilde{C}_n e^{inj2\pi/N}}{k\left(1 - r_n^2 + i2\xi r_n\right)} \tag{75}$$

Dynamic Response of "Water Tank Structure" by FFT.

The dynamic response $y(t_j)$ in frequency domain of a general SDOF system (such as the "water tank structure") can be obtained by Equation (75), and the required coefficients $\widetilde{c}_n$ can be computed by Equation (71). Both of these equations can be represented (except for the sign), by the following general exponential function

$$A(j) = factor * \sum_{n=0}^{N-1} A^{(0)}(n) W^{jn} \tag{76}$$

where

$$W = e^{sign * i2\pi/N} \tag{77}$$

If Equation (71) needs be computed for $\widetilde{C}_n$, then one should define $factor = \frac{1}{N}$, sign = -1, and $A^{(0)} = F(t_j)$. However, if Equation (75) needs be computed for $y(t_j)$, then one should define $factor = 1$, sign = +1, and $A^{(0)} = \frac{\widetilde{C}_n}{k\left(1 - r_n^2 + i2\xi r_n\right)}$.

It is important to notice that Equation (76) has the same form as shown in the earlier Equation (74). However, the definition of W in Equation (77) is different from the one shown in Equation (4, Ch. 11.05) by a negative sign in the power of W. Therefore, efficient

FFT subroutine (with user's specified SIGN = 1, or -1) can be utilized, as given at
http://numericalmethods.eng.usf.edu/simulations/mtl/11fft/fft_civil_engg_example12.m

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