Differentiation-Continuous Functions

Computer Engineering Majors

Authors: Autar Kaw, Sri Harsha Garapati

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Differentiation – Continuous Functions

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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite $'\Delta x'$

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

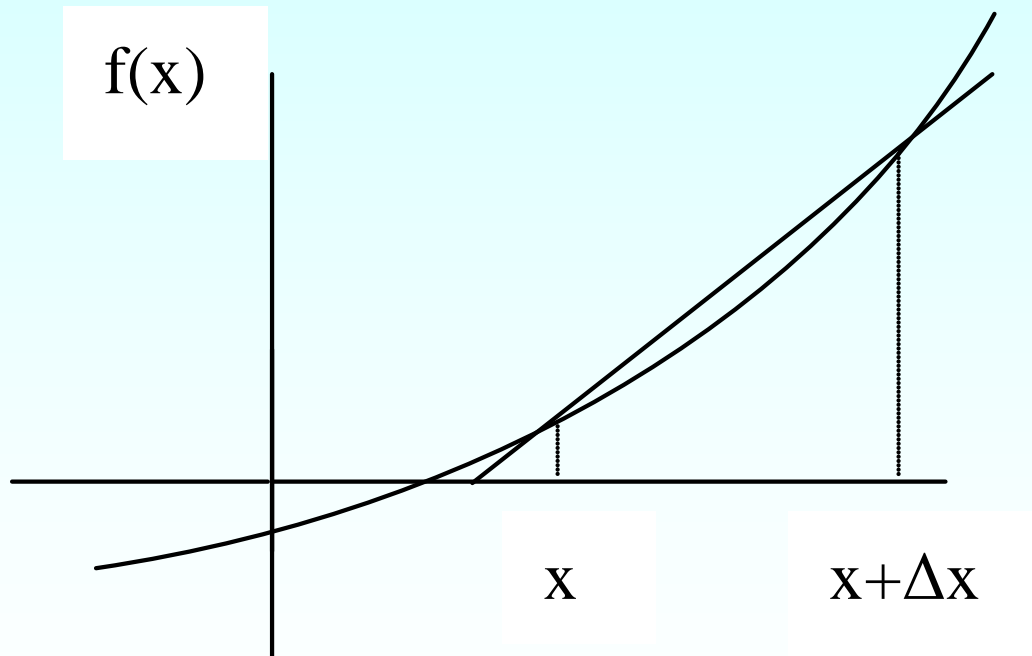# Graphical Representation Of Forward Difference Approximation

Figure 1 Graphical Representation of forward difference approximation of first derivative.

Example 1

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

a) Use forward divided difference approximation of the first derivative of $f(x)$ to calculate its derivative at $x = 0.1$ for $a = 0.24.$ Use a step size of $\Delta x = 0.05$. Also calculate the absolute relative true error.

b) Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also calculate the relative true error.

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Example 1 Cont.

Solution:

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i+1} = x + \Delta x$$
$$= 0.1 + 0.05$$
$$= 0.15$$

$$f(0.1) = 1 - e^{(-0.24 \times 0.1)}$$
$$= 0.023714$$

$$f(0.15) = 1 - e^{(-0.24 \times 0.15)}$$
$$= 0.035360$$

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Example 1 Cont.

$$f^{'}(0.1) \approx \frac{f(0.15) - f(0.1)}{0.05}$$

$$= \frac{0.035360 - 0.023714}{0.05}$$

$$= 0.23291$$

The exact value of $f^{'}(0.1)$ can be calculated by differentiating

$$f(x) = 1 - e^{-ax}, x \geq 0$$

as

$$f^{'}(x) = \frac{d}{dx}[f(x)]$$

Example 1 Cont.

Knowing that

$$\frac{d}{dx}\left[e^{-ax}\right] = -ae^{-ax}$$

we find

$$f'(x) = \frac{d}{dx}(1 - e^{-ax})$$
$$= ae^{-ax}$$
$$= 0.24e^{-0.24x}$$

$$f'(0.1) = (0.24)(e^{-(0.24 \times 0.1)}$$
$$= 0.23431$$

Example 1 Cont.

The absolute relative true error is

$$\left|\in_t\right| = \left|\frac{\text{True Value - Approximate Value}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{0.23431 - 0.23291}{0.23431}\right| \times 100$$

$$= 0.59761\%$$

Example 1 Cont.

b)

$$a = 0.12$$

$$f(0.1) = 1 - e^{-(0.12 \times 0.1)}$$
$$= 0.011928$$

$$f(0.15) = 1 - e^{-(0.12 \times 0.15)}$$
$$= 0.017839$$

$$f'(0.1) \approx \frac{f(0.15) - f(0.1)}{0.05}$$

$$\approx \frac{0.017838 - 0.011928}{0.05}$$

$$\approx 0.11821$$

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Example 1 Cont.

$$f'(x) = \frac{d}{dx}(1 - e^{-ax})$$

$$= ae^{-ax}$$

$$= 0.12e^{-0.12x}$$

$$f'(0.1) = (0.12)(e^{-(0.12 \times 0.1)}$$

$$= 0.11856$$

Example 1 Cont.

The absolute relative true error is

$$\left| \in_t \right| = \left| \frac{\text{True Value - Approximate Value}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.11857 - 0.11821}{0.11857} \right| \times 100$$

$$= 0.29940\%$$

The estimate of the derivative decreased.

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Backward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx' ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If 'Δx' is chosen as a negative number,

$$f'(x) \approx \frac{f(x - \Delta x) - f(x)}{-\Delta x}$$

$$= \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

Backward Difference Approximation of the First Derivative Cont.

This is a backward difference approximation as you are taking a point backward from x. To find the value of $f'(x)$ at $x = x_i$, we may choose another point 'Δx' behind as $x = x_{i-1}$. This gives

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

where

$$\Delta x = x_i - x_{i-1}$$

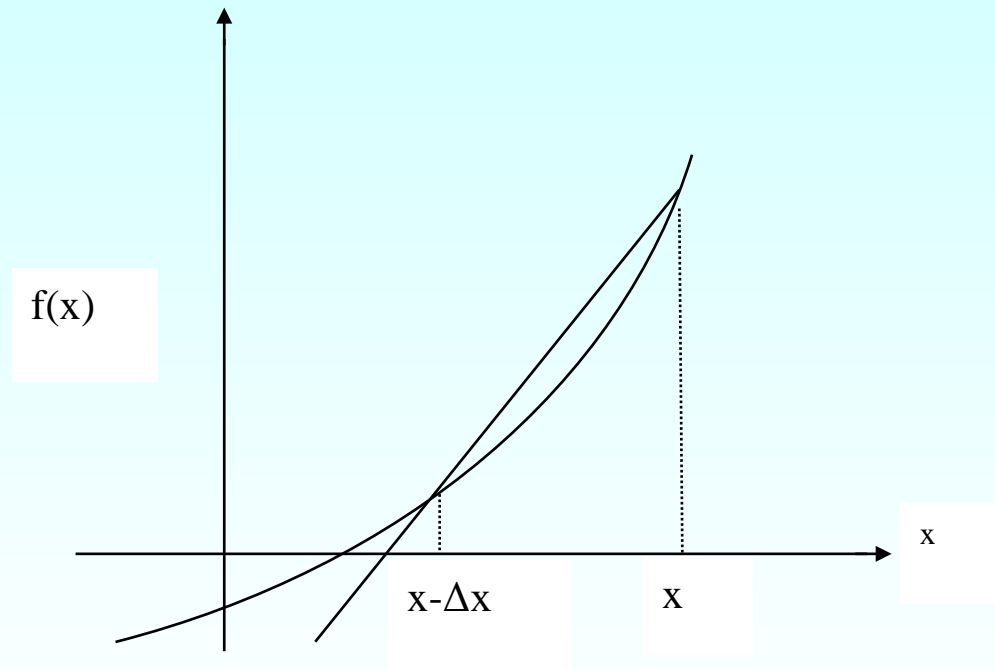# Backward Difference Approximation of the First Derivative Cont.

Figure 2 Graphical Representation of backward difference approximation of first derivative

Example 2

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

a) Use backward divided difference approximation of the first derivative of $f(x)$ to calculate its derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also calculate the absolute relative true error.

b) Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also calculate the relative true error.

Example 2 Cont.

Solution

a) $f(x_i) \approx \dfrac{f(x_i) - f(x_{i-1})}{\Delta x}$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i-1} = x_i - \Delta x$$
$$= 0.1 - 0.05$$
$$= 0.05$$

Example 2 Cont.

$$f(0.1) = 1 - e^{-(0.24 \times 0.1)}$$

$$= 0.023714$$

$$f(0.05) = 1 - e^{-(0.24 \times 0.05)}$$

$$= 0.011928$$

$$f'(0.1) \approx \frac{f(0.1) - f(0.05)}{0.05}$$

$$\approx \frac{0.023714 - 0.011928}{0.05}$$

$$\approx 0.23572$$

Example 2 Cont.

The absolute relative true error is

$$\left|\epsilon_t\right| = \left|\frac{\text{True Value - Approximate Value}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{0.23431 - 0.23572}{0.23431}\right| \times 100$$

$$= 0.60241\%$$

Example 2 Cont.

b) $a = 0.12$

$f(0.1) = 1 - e^{-(0.12 \times 0.1)}$

$\qquad = 0.011928$

$f(0.05) = 1 - e^{-(0.12 \times 0.05)}$

$\qquad = 0.0059820$

$f'(0.1) \approx \dfrac{f(0.1) - f(0.05)}{0.05}$

$\qquad \approx \dfrac{0.011928 - 0.0059820}{0.05}$

$\qquad \approx 0.11893$

Example 2 Cont.

The absolute relative true error is

$$\left| \in_t \right| = \left| \frac{\text{True Value - Approximate Value}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.11857 - 0.11893}{0.11857} \right| \times 100$$

$$= 0.30060\%$$

The estimate of the derivative decreased.

Derive the forward difference approximation from Taylor series

Taylor's theorem says that if you know the value of a function $'f'$ at a point x_i and all its derivatives at that point, provided the derivatives are continuous between x_i and x_{i+1}, then

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \ldots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) + \ldots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + 0(\Delta x)$$

Derive the forward difference approximation from Taylor series Cont.

The $(0\Delta x)$ term shows that the error in the approximation is of the order of (Δx) Can you now derive from Taylor series the formula for backward divided difference approximation of the first derivative?

As shown above, both forward and backward divided difference approximation of the first derivative are accurate on the order of $(0\Delta x)$ Can we get better approximations? Yes, another method to approximate the first derivative is called the **Central difference approximation of the first derivative**.

Derive the forward difference approximation from Taylor series Cont.

From Taylor series

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

Subtracting equation (2) from equation (1)

$$f(x_{i+1}) - f(x_{i-1}) = f'(x_i)(2\Delta x) + \frac{2f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} - \frac{f'''(x_i)}{3!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + 0(\Delta x)^2$$

Central Divided Difference

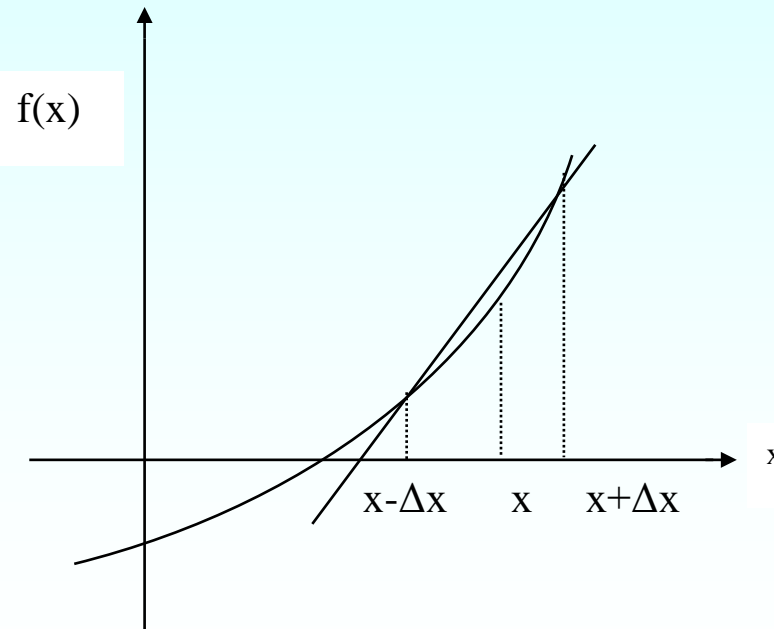Hence showing that we have obtained a more accurate formula as the error is of the order of $0(\Delta x)^2$.

Figure 3 Graphical Representation of central difference approximation of first derivative

Example 3

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

a) Use central divided difference approximation of the first derivative of $f(x)$ to calculate its derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also calculate the absolute relative true error.

b) Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also calculate the relative true error.

Example 3 cont.

Solution

a) $f'(x_i) \approx \dfrac{f(t_{x+1}) - f(t_{x-1})}{2\Delta x}$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i+1} = x_i + \Delta x$$

$$= 0.1 + 0.05$$

$$= 0.15$$

$$x_{i-1} = x_i - \Delta x$$

$$= 0.1 - 0.05$$

$$= 0.05$$

Example 3 cont.

$$f'(0.1) \approx \frac{f(0.15) - f(0.05)}{2(0.05)}$$

$$\approx \frac{f(0.15) - f(0.05)}{0.1}$$

$$f(0.15) = 1 - e^{-(0.24 \times 0.15)}$$

$$= 0.035360$$

$$f(0.05) = 1 - e^{-(0.24 \times 0.05)}$$

$$= 0.011928$$

Example 3 cont.

$$f^{'}(0.1) \approx \frac{f(0.15) - f(0.05)}{0.1}$$

$$\approx \frac{0.035360 - 0.011928}{0.1}$$

$$\approx 0.23431$$

The absolute relative true error is

$$\left|\in_{t}\right| = \left|\frac{\text{True Value - Approximate Value}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{0.23431 - 0.23431}{0.23431}\right| \times 100$$

$$= 0.0024000\%$$

Example 3 cont.

b) $a = 0.12$

$$f(0.15) = 1 - e^{-(0.12 \times 0.15)}$$

$$= 0.017839$$

$$f(0.05) = 1 - e^{-(0.12 \times 0.05)}$$

$$= 0.0059820$$

$$f'(0.1) \approx \frac{f(0.15) - f(0.05)}{0.1}$$

$$\approx \frac{0.017839 - 0.0059820}{0.1}$$

$$\approx 0.11857$$

Example 3 cont.

The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Value - Approximate Value}}{\text{True Value}} \right| x100$$

$$= \left| \frac{0.11857 - 0.11857}{0.11857} \right| x100$$

$$= 6.0000 \times 10^{-4}\%$$

The results from the three difference approximations are given in Table 1.

Comparision

Table 1 Summary of $f'(0.1)$ using different divided difference approximations

Type of Difference Approximation	$f'(0.1), a = 0.24$	$\left\|\in_t\right\|\%, a = 0.24$	$f'(0.1), a = 0.12$	$\left\|\in_t\right\|\%, a = 0.12$
Forward	0.23291	0.59761	0.11821	0.29940
Backward	0.23572	0.60241	0.11893	0.30060
Central	0.23431	0.0024000	0.11857	6.0000×10^{-4}

Finding the value of the derivative within a prespecified tolerance

In real life, one would not know the exact value of the derivative – so how would one know how accurately they have found the value of the derivative.

A simple way would be to start with a step size and keep on halving the step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.

Take the example of finding $v'(t)$ for

$$v(t) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t$$

at $t = 16$ using the backward divided difference scheme.

Finding the value of the derivative within a prespecified tolerance Cont.

Given in Table 2 are the values obtained using the backward difference approximation method and the corresponding absolute relative approximate errors.

Table 2 First derivative approximations and relative errors for different Δt values of backward difference scheme

Δt	$v'(t)$	$\left|\in_a\right|\%$
2	28.915	
1	29.289	1.2792
0.5	29.480	0.64787
0.25	29.577	0.32604
0.125	29.625	0.16355

Finding the value of the derivative within a prespecified tolerance Cont.

From the above table, one can see that the absolute relative approximate error decreases as the step size is reduced. At $\Delta t = 0.125$ the absolute relative approximate error is 0.16355%, meaning that at least 2 significant digits are correct in the answer.

Finite Difference Approximation of Higher Derivatives

One can use Taylor series to approximate a higher order derivative.

For example, to approximate $f''(x)$, the Taylor series for

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f'''(x_i)}{3!}(2\Delta x)^3 + \dots$$

where

$$x_{i+2} = x_i + 2\Delta x$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 \dots$$

where

$$x_{i-1} = x_i - \Delta x$$

Finite Difference Approximation of Higher Derivatives Cont.

Subtracting 2 times equation (4) from equation (3) gives

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)(\Delta x)^2 + f'''(x_i)(\Delta x)^3 \dots$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} - f'''(x_i)(\Delta x) + \dots$$

$$f''(x_i) \cong \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} + 0(\Delta x) \qquad\qquad (5)$$

Example 4

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4-2100t}\right] - 9.8t, 0 \le t \le 30$$

Use forward difference approximation of the second derivative $v(t)$ of to calculate the jerk at $t = 16s$. Use a step size of $\Delta t = 2s$.

Example 4 Cont.

Solution

$$j(t_i) \approx \frac{v(t_{i+2}) - 2v(t_{i+1}) + v(t_i)}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i+2} &= t_i + 2(\Delta t) \\ &= 16 + 2(2) \\ &= 20 \end{aligned}$$

$$j(16) \approx \frac{v(20) - 2v(18) + v(16)}{(2)^2}$$

Example 4 Cont.

$$v(20) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(20)} \right] - 9.8(20)$$

$$= 517.35 \text{m/s}$$

$$v(18) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18)$$

$$= 453.02 m/s$$

$$v(16) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16)$$

$$= 392.07 \text{m/s}$$

Example 4 Cont.

$$j(16) \approx \frac{517.35 - 2(453.02) + 392.07}{4}$$

$$\approx 0.84515 \text{m/s}^3$$

The exact value of $j(16)$ can be calculated by differentiating

$$v(t) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t}\right] - 9.8t$$

twice as

$$a(t) = \frac{d}{dt}[v(t)] \quad \text{and} \quad j(t) = \frac{d}{dt}[a(t)]$$

Example 4 Cont.

Knowing that

$$\frac{d}{dt}\big[\ln(t)\big] = \frac{1}{t} \quad \text{and} \quad \frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$a(t) = 2000\left(\frac{14 \times 10^4 - 2100t}{14 \times 10^4}\right)\frac{d}{dt}\left(\frac{14 \times 10^4}{14 \times 10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14 \times 10^4 - 2100t}{14 \times 10^4}\right)(-1)\left(\frac{14 \times 10^4}{\left(14 \times 10^4 - 2100t\right)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

Example 4 Cont.

Similarly it can be shown that

$$j(t) = \frac{d}{dt}[a(t)]$$

$$= \frac{18000}{(-200 + 3t)^2}$$

$$j(16) = \frac{18000}{[-200 + 3(16)]^2}$$

$$= 0.77909 \, \text{m/s}^3$$

The absolute relative true error is

$$\left| \in_t \right| = \left| \frac{0.77909 - 0.84515}{0.77909} \right| \times 100$$

$$= 8.4797 \, \%$$

Higher order accuracy of higher order derivatives

The formula given by equation (5) is a forward difference approximation of

the second derivative and has the error of the order of (Δx) . Can we get

a formula that has a better accuracy? We can get the central difference

approximation of the second derivative.

The Taylor series for

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f''''(x_i)}{4!}(\Delta x)^4 \ldots \qquad (6)$$

where

$$x_{i+1} = x_i + \Delta x$$

Higher order accuracy of higher order derivatives Cont.

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f''''(x_i)}{4!}(\Delta x)^4 \dots \qquad (7)$$

where

$$x_{i-1} = x_i - \Delta x$$

Adding equations (6) and (7), gives

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)(\Delta x)^2 + f'''(x_i)\frac{(\Delta x)^4}{12}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} - \frac{f'''(x_i)(\Delta x)^2}{12}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} + 0(\Delta x)^2$$

Example 5

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, 0 \le t \le 30$$

Use central difference approximation of second derivative of $v(t)$ to calculate the jerk at $t = 16s$. Use a step size of $\Delta t = 2s$.

Example 5 Cont.

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - 2v(t_i) + v(t_{i-1})}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$
$$= 16 + 2$$
$$= 18$$

$$t_{i-1} = t_i - \Delta t$$
$$= 16 - 2$$
$$= 14$$

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$

Example 5 Cont.

$$v(18) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18)$$

$$= 453.02 \, \text{m/s}$$

$$v(16) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16)$$

$$= 392.07 \, \text{m/s}$$

$$v(14) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(14)} \right] - 9.8(14)$$

$$= 334.24 \, \text{m/s}$$

Example 5 Cont.

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$

$$\approx \frac{453.02 - 2(392.07) + 334.24}{4}$$

$$\approx 0.77969 \text{m/s}^3$$

The absolute relative true error is

$$\left| \in_t \right| = \left| \frac{0.77908 - 0.78}{0.77908} \right| \times 100$$

$$= 0.077992\%$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/continuous_02dif.html

THE END

http://numericalmethods.eng.usf.edu