# Chapter 04.05
# System of Equations

*After reading this chapter, you should be able to:*

1. *setup simultaneous linear equations in matrix form and vice-versa,*
2. *understand the concept of the inverse of a matrix,*
3. *know the difference between a consistent and inconsistent system of linear equations, and*
4. *learn that a system of linear equations can have a unique solution, no solution or infinite solutions.*

**Matrix algebra is used for solving systems of equations. Can you illustrate this concept?**

Matrix algebra is used to solve a system of simultaneous linear equations. In fact, for many mathematical procedures such as the solution to a set of nonlinear equations, interpolation, integration, and differential equations, the solutions reduce to a set of simultaneous linear equations. Let us illustrate with an example for interpolation.

**Example 1**

The upward velocity of a rocket is given at three different times on the following table.

**Table 5.1.** Velocity vs. time data for a rocket

| Time, $t$ | Velocity, $v$ |
|---|---|
| (s) | (m/s) |
| 5 | 106.8 |
| 8 | 177.2 |
| 12 | 279.2 |

The velocity data is approximated by a polynomial as

$$v(t) = at^2 + bt + c, \quad 5 \le t \le 12.$$

Set up the equations in matrix form to find the coefficients $a, b, c$ of the velocity profile.

**Solution**

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2)$, and $(t_3, v_3)$ where from table 5.1.

$$t_1 = 5, v_1 = 106.8$$

$$t_2 = 8, v_2 = 177.2$$

$$t_3 = 12, v_3 = 279.2$$

Requiring that $v(t) = at^2 + bt + c$ passes through the three data points gives

$$v(t_1) = v_1 = at_1^2 + bt_1 + c$$
$$v(t_2) = v_2 = at_2^2 + bt_2 + c$$
$$v(t_3) = v_3 = at_3^2 + bt_3 + c$$

Substituting the data $(t_1, v_1), (t_2, v_2),$ and $(t_3, v_3)$ gives

$$a(5^2) + b(5) + c = 106.8$$
$$a(8^2) + b(8) + c = 177.2$$
$$a(12^2) + b(12) + c = 279.2$$

or

$$25a + 5b + c = 106.8$$
$$64a + 8b + c = 177.2$$
$$144a + 12b + c = 279.2$$

This set of equations can be rewritten in the matrix form as

$$\begin{bmatrix} 25a + & 5b + & c \\ 64a + & 8b + & c \\ 144a + & 12b + & c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above equation can be written as a linear combination as follows

$$a\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + b\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

and further using matrix multiplication gives

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above is an illustration of why matrix algebra is needed. The complete solution to the set of equations is given later in this chapter.

A general set of $m$ linear equations and $n$ unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots.. + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

Denoting the matrices by $[A]$, $[X]$, and $[C]$, the system of equation is $[A][X]=[C]$, where $[A]$ is called the coefficient matrix, $[C]$ is called the right hand side vector and $[X]$ is called the solution vector.

Sometimes $[A][X]=[C]$ systems of equations are written in the augmented form. That is

$$[A \vdots C] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots\ldots & a_{1n} & c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & c_2 \\ \vdots & & & \vdots & \\ \vdots & & & \vdots & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & c_n \end{array}\right]$$

**A system of equations can be consistent or inconsistent. What does that mean?**

A system of equations $[A][X]=[C]$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, a consistent system of equations does not mean a unique solution, that is, a consistent system of equations may have a unique solution or infinite solutions (Figure 1).

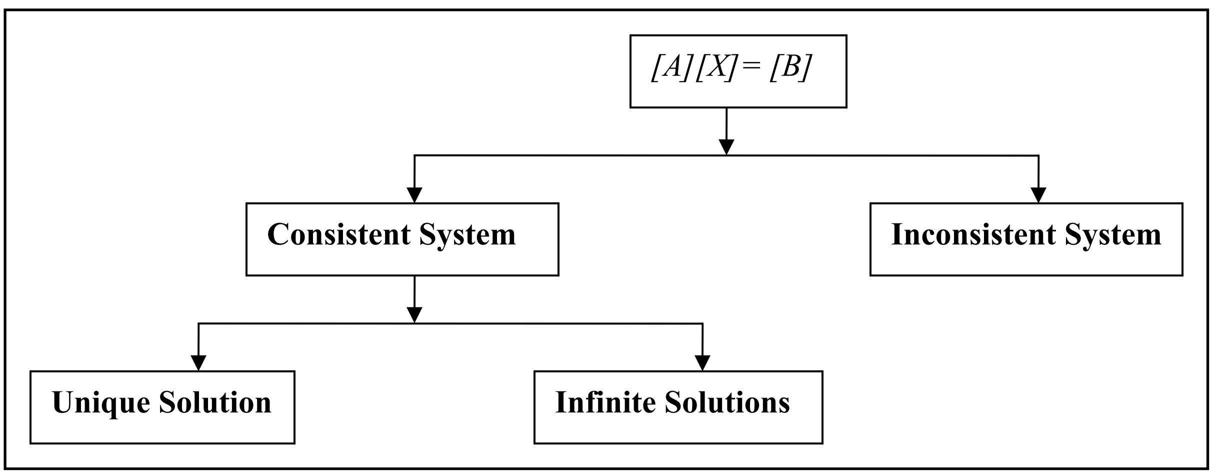


**Figure 5.1.** Consistent and inconsistent system of equations flow chart.

**Example 2**

Give examples of consistent and inconsistent system of equations.

**Solution**

a) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is a consistent system of equations as it has a unique solution, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

b) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

is also a consistent system of equations but it has infinite solutions as given as follows. Expanding the above set of equations,

$$2x + 4y = 6$$

$$x + 2y = 3$$

you can see that they are the same equation. Hence, any combination of $(x, y)$ that satisfies

$$2x + 4y = 6$$

is a solution. For example $(x, y) = (1,1)$ is a solution. Other solutions include $(x, y) = (0.5,1.25)$, $(x, y) = (0,\ 1.5)$, and so on.

c) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is inconsistent as no solution exists.

**How can one distinguish between a consistent and inconsistent system of equations?**

A system of equations $[A][X] = [C]$ is *consistent* if the rank of $A$ is equal to the rank of the augmented matrix $[A \vdots C]$

A system of equations $[A][X] = [C]$ is *inconsistent* if the rank of $A$ is less than the rank of the augmented matrix $[A \vdots C]$.

But, what do you mean by rank of a matrix?

The rank of a matrix is defined as the order of the largest square submatrix whose determinant is not zero.

**Example 3**

What is the rank of

$$[A] = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}?$$

**Solution**

The largest square submatrix possible is of order 3 and that is $[A]$ itself. Since $\det(A) = -23 \neq 0,$ the rank of $[A] = 3.$

**Example 4**

What is the rank of

$$[A]=\begin{bmatrix}3 & 1 & 2\\ 2 & 0 & 5\\ 5 & 1 & 7\end{bmatrix}?$$

**Solution**

The largest square submatrix of $[A]$ is of order 3 and that is $[A]$ itself. Since $\det(A)=0$, the rank of $[A]$ is less than 3. The next largest square submatrix would be a $2\times 2$ matrix. One of the square submatrices of $[A]$ is

$$[B]=\begin{bmatrix}3 & 1\\ 2 & 0\end{bmatrix}$$

and $\det(B)=-2\neq 0$. Hence the rank of $[A]$ is 2. There is no need to look at other $2\times 2$ submatrices to establish that the rank of $[A]$ is 2.

**Example 5**

How do I now use the concept of rank to find if

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}106.8\\ 177.2\\ 279.2\end{bmatrix}$$

is a consistent or inconsistent system of equations?

**Solution**

The coefficient matrix is

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

and the right hand side vector is

$$[C]=\begin{bmatrix}106.8\\ 177.2\\ 279.2\end{bmatrix}$$

The augmented matrix is

$$[B]=\begin{bmatrix}25 & 5 & 1 & \vdots & 106.8\\ 64 & 8 & 1 & \vdots & 177.2\\ 144 & 12 & 1 & \vdots & 279.2\end{bmatrix}$$

Since there are no square submatrices of order 4 as $[B]$ is a $3\times 4$ matrix, the rank of $[B]$ is at most 3. So let us look at the square submatrices of $[B]$ of order 3; if any of these square submatrices have determinant not equal to zero, then the rank is 3. For example, a submatrix of the augmented matrix $[B]$ is

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

has $\det(D) = -84 \neq 0$.

Hence the rank of the augmented matrix $[B]$ is 3. Since $[A] = [D]$, the rank of $[A]$ is 3. Since the rank of the augmented matrix $[B]$ equals the rank of the coefficient matrix $[A]$, the system of equations is consistent.

**Example 6**

Use the concept of rank of matrix to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

is consistent or inconsistent?

**Solution**

The coefficient matrix is given by

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

and the right hand side

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

The augmented matrix is

$$[B] = \begin{bmatrix} 25 & 5 & 1 & :106.8 \\ 64 & 8 & 1 & :177.2 \\ 89 & 13 & 2 & :284.0 \end{bmatrix}$$

Since there are no square submatrices of order 4 as $[B]$ is a $4 \times 3$ matrix, the rank of the augmented $[B]$ is at most 3. So let us look at square submatrices of the augmented matrix $[B]$ of order 3 and see if any of these have determinants not equal to zero. For example, a square submatrix of the augmented matrix $[B]$ is

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

has $\det(D) = 0$. This means, we need to explore other square submatrices of order 3 of the augmented matrix $[B]$ and find their determinants.

That is,

$$[E]=\begin{bmatrix}5 & 1 & 106.8\\ 8 & 1 & 177.2\\ 13 & 2 & 284.0\end{bmatrix}$$

$\det(E)=0$

$$[F]=\begin{bmatrix}25 & 5 & 106.8\\ 64 & 8 & 177.2\\ 89 & 13 & 284.0\end{bmatrix}$$

$\det(F)=0$

$$[G]=\begin{bmatrix}25 & 1 & 106.8\\ 64 & 1 & 177.2\\ 89 & 2 & 284.0\end{bmatrix}$$

$\det(G)=0$

All the square submatrices of order $3\times 3$ of the augmented matrix $[B]$ have a zero determinant. So the rank of the augmented matrix $[B]$ is less than 3. Is the rank of augmented matrix $[B]$ equal to 2?. One of the $2\times 2$ submatrices of the augmented matrix $[B]$ is

$$[H]=\begin{bmatrix}25 & 5\\ 64 & 8\end{bmatrix}$$

and

$$\det(H)=-120\neq 0$$

So the rank of the augmented matrix $[B]$ is 2.

Now we need to find the rank of the coefficient matrix $[B]$.

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}$$

and

$$\det(A)=0$$

So the rank of the coefficient matrix $[A]$ is less than 3. A square submatrix of the coefficient matrix $[A]$ is

$$[J]=\begin{bmatrix}5 & 1\\ 8 & 1\end{bmatrix}$$

$$\det(J)=-3\neq 0$$

So the rank of the coefficient matrix $[A]$ is 2.

Hence, rank of the coefficient matrix $[A]$ equals the rank of the augmented matrix $[B]$. So the system of equations $[A][X]=[C]$ is consistent.

**Example 7**

Use the concept of rank to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 280.0 \end{bmatrix}$$

is consistent or inconsistent.

**Solution**

The augmented matrix is

$$[B] = \begin{bmatrix} 25 & 5 & 1 & :106.8 \\ 64 & 8 & 1 & :177.2 \\ 89 & 13 & 2 & :280.0 \end{bmatrix}$$

Since there are no square submatrices of order $4\times 4$ as the augmented matrix $[B]$ is a $4\times 3$ matrix, the rank of the augmented matrix $[B]$ is at most 3. So let us look at square submatrices of the augmented matrix ($B$) of order 3 and see if any of the $3\times 3$ submatrices have a determinant not equal to zero. For example, a square submatrix of order $3\times 3$ of $[B]$

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

det($D$) = 0

So it means, we need to explore other square submatrices of the augmented matrix $[B]$

$$[E] = \begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$$

$$\det(E0 \neq 12.0 \neq 0.$$

So the rank of the augmented matrix $[B]$ is 3.

The rank of the coefficient matrix $[A]$ is 2 from the previous example.

Since the rank of the coefficient matrix $[A]$ is less than the rank of the augmented matrix $[B]$, the system of equations is inconsistent. Hence, no solution exists for $[A][X]=[C]$.

**If a solution exists, how do we know whether it is unique?**

In a system of equations $[A][X]=[C]$ that is consistent, the rank of the coefficient matrix $[A]$ is the same as the augmented matrix $[A|C]$. If in addition, the rank of the coefficient matrix $[A]$ is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix $[A]$ is less than the number of unknowns, then infinite solutions exist.

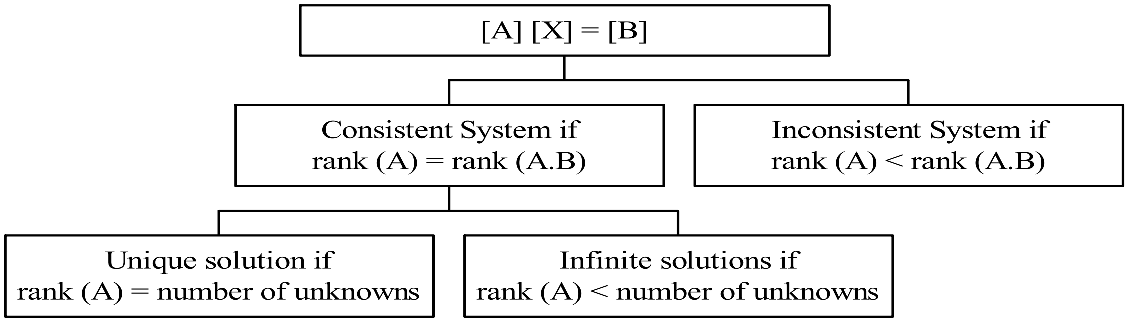


**Figure 5.2.** Flow chart of conditions for consistent and inconsistent system of equations.

**Example 8**

We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

is a consistent system of equations. Does the system of equations have a unique solution or does it have infinite solutions?

**Solution**

The coefficient matrix is

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and the right hand side is

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

While finding out whether the above equations were consistent in an earlier example, we found that the rank of the coefficient matrix ($A$) equals rank of augmented matrix $[A \vdots C]$ equals 3.

The solution is unique as the number of unknowns = 3 = rank of ($A$).

**Example 9**

We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

is a consistent system of equations. Is the solution unique or does it have infinite solutions.

**Solution**

While finding out whether the above equations were consistent, we found that the rank of the coefficient matrix $[A]$ equals the rank of augmented matrix $(A \vdots C)$ equals 2

Since the rank of $[A] = 2 <$ number of unknowns = 3, infinite solutions exist.

**If we have more equations than unknowns in [A] [X] = [C], does it mean the system is inconsistent?**

No, it depends on the rank of the augmented matrix $[A \vdots C]$ and the rank of $[A]$.

a) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 284.0 \end{bmatrix}$$

is consistent, since

rank of augmented matrix = 3
rank of coefficient matrix = 3

Now since the rank of $(A)$ = 3 = number of unknowns, the solution is not only consistent but also unique.

b) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 280.0 \end{bmatrix}$$

is inconsistent, since

rank of augmented matrix = 4
rank of coefficient matrix = 3

c) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 50 & 10 & 2 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 213.6 \\ 280.0 \end{bmatrix}$$

is consistent, since

rank of augmented matrix = 2
rank of coefficient matrix = 2

But since the rank of $[A]$ = 2 < the number of unknowns = 3, infinite solutions exist.

**Consistent systems of equations can only have a unique solution or infinite solutions. Can a system of equations have more than one but not infinite number of solutions?**

No, you can only have either a unique solution or infinite solutions. Let us suppose $[A][X] = [C]$ has two solutions $[Y]$ and $[Z]$ so that

$$[A][Y]=[C]$$
$$[A][Z]=[C]$$

If $r$ is a constant, then from the two equations

$$r[A][Y]=r[C]$$
$$(1-r)[A][Z]=(1-r)[C]$$

Adding the above two equations gives

$$r[A][Y]+(1-r)[A][Z]=r[C]+(1-r)[C]$$
$$[A](r[Y]+(1-r)[Z])=[C]$$

Hence

$$r[Y]+(1-r)[Z]$$

is a solution to

$$[A][X]=[C]$$

Since $r$ is any scalar, there are infinite solutions for $[A][X]=[C]$ of the form

$$r[Y]+(1-r)[Z]$$

**Can you divide two matrices?**

If $[A][B]=[C]$ is defined, it might seem intuitive that $[A]=\frac{[C]}{[B]}$, but matrix division is not defined like that. However an inverse of a matrix can be defined for certain types of square matrices. The inverse of a square matrix $[A]$, if existing, is denoted by $[A]^{-1}$ such that

$$[A][A]^{-1}=[I]=[A]^{-1}[A]$$

Where $[I]$ is the identity matrix.

In other words, let $[A]$ be a square matrix. If $[B]$ is another square matrix of the same size such that $[B][A]=[I]$, then $[B]$ is the inverse of $[A]$. $[A]$ is then called to be invertible or nonsingular. If $[A]^{-1}$ does not exist, $[A]$ is called noninvertible or singular.

If $[A]$ and $[B]$ are two $n\times n$ matrices such that $[B][A]=[I]$, then these statements are also true

- $[B]$ is the inverse of $[A]$
- $[A]$ is the inverse of $[B]$
- $[A]$ and $[B]$ are both invertible
- $[A]$ $[B]$=$[I]$.
- $[A]$ and $[B]$ are both nonsingular
- all columns of $[A]$ and $[B]$are linearly independent
- all rows of $[A]$ and $[B]$ are linearly independent.

**Example 10**

Determine if

$$[B]=\begin{bmatrix}3 & 2\\ 5 & 3\end{bmatrix}$$

is the inverse of

$$[A]=\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}$$

**Solution**

$$[B][A]=\begin{bmatrix}3 & 2\\ 5 & 3\end{bmatrix}\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=[I]$$

Since

$$[B][A]=[I],$$

$[B]$ is the inverse of $[A]$ and $[A]$ is the inverse of $[B]$.

But, we can also show that

$$[A][B]=\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}\begin{bmatrix}3 & 2\\ 5 & 3\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=[I]$$

to show that $[A]$ is the inverse of $[B]$.

**Can I use the concept of the inverse of a matrix to find the solution of a set of equations [A] [X] = [C]?**

Yes, if the number of equations is the same as the number of unknowns, the coefficient matrix $[A]$ is a square matrix.

Given

$$[A][X]=[C]$$

Then, if $[A]^{-1}$ exists, multiplying both sides by $[A]^{-1}$.

$$[A]^{-1}[A][X]=[A]^{-1}[C]$$
$$[I][X]=[A]^{-1}[C]$$
$$[X]=[A]^{-1}[C]$$

This implies that if we are able to find $[A]^{-1}$, the solution vector of $[A][X]=[C]$ is simply a multiplication of $[A]^{-1}$ and the right hand side vector, $[C]$.

**How do I find the inverse of a matrix?**

If $[A]$ is a $n\times n$ matrix, then $[A]^{-1}$ is a $n\times n$ matrix and according to the definition of inverse of a matrix

$$[A][A]^{-1}=[I]$$

Denoting

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix}$$

$$[A]^{-1}=\begin{bmatrix} a'_{11} & a'_{12} & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & a'_{nn} \end{bmatrix}$$

$$[I]=\begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & & & & 0 \\ 0 & & \cdot & & & \cdot \\ \cdot & & & 1 & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Using the definition of matrix multiplication, the first column of the $[A]^{-1}$ matrix can then be found by solving

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix}\begin{bmatrix} a'_{11} \\ a'_{21} \\ \cdot \\ \cdot \\ a'_{n1} \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Similarly, one can find the other columns of the $[A]^{-1}$ matrix by changing the right hand side accordingly.

**Example 11**

The upward velocity of the rocket is given by

**Table 5.2.** Velocity vs time data for a rocket

| Time, $t$ (s) | Velocity, $v$ (m/s) |
|---|---|
| 5 | 106.8 |
| 8 | 177.2 |
| 12 | 279.2 |

In an earlier example, we wanted to approximate the velocity profile by

$$v(t)=at^2+bt+c, \quad 5\le t\le 12$$

We found that the coefficients $a, b, \text{and}\, c$ in $v(t)$ are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} \text{a} \\ \text{b} \\ \text{c} \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

First, find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and then use the definition of inverse to find the coefficients $a, b,$ and $c$.

**Solution**

If

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$$

is the inverse of $[A]$, then

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives three sets of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving the above three sets of equations separately gives

$$\begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

Now

$$[A][X] = [C]$$

where

$$[X] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the definition of $[A]^{-1}$,

$$[A]^{-1}[A][X] = [A]^{-1}[C]$$

$$[\mathrm{X}] = [A]^{-1}[C]$$

$$\begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix} \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Hence

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2905 \\ 19.69 \\ 1.086 \end{bmatrix}$$

So

$$v(t) = 0.2905t^2 + 19.69t + 1.086, 5 \le t \le 12$$

**Is there another way to find the inverse of a matrix?**

For finding the inverse of small matrices, the inverse of an invertible matrix can be found by

$$[A]^{-1} = \frac{1}{\det(A)} adj(A)$$

where

$$adj(A)=\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{\mathrm{T}}$$

where $C_{ij}$ are the cofactors of $a_{ij}$. The matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & \cdots & \cdots & C_{nn} \end{bmatrix}$$

itself is called the matrix of cofactors from $[A]$. Cofactors are defined in Chapter 4.

**Example 12**

Find the inverse of

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

**Solution**

From Example 4.6 in Chapter 4, we found

$$\det(A)=-84$$

Next we need to find the adjoint of $[A]$. The cofactors of $A$ are found as follows.

The minor of entry $a_{11}$ is

$$M_{11}=\begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 8 & 1 \\ 12 & 1 \end{vmatrix}$$

$$=-4$$

The cofactors of entry $a_{11}$ is

$$C_{11}=(-1)^{1+1}M_{11}$$

$$=M_{11}$$

$$=-4$$

The minor of entry $a_{12}$ is

$$M_{12}=\begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 64 & 1 \\ 144 & 1 \end{vmatrix}$$
$$= -80$$

The cofactor of entry $a_{12}$ is

$$C_{12} = (-1)^{1+2} M_{12}$$
$$= -M_{12}$$
$$= -(-80)$$
$$= 80$$

Similarly

$$C_{13} = -384$$
$$C_{21} = 7$$
$$C_{22} = -119$$
$$C_{23} = 420$$
$$C_{31} = -3$$
$$C_{32} = 39$$
$$C_{33} = -120$$

Hence the matrix of cofactors of $[A]$ is

$$[C] = \begin{bmatrix} -4 & 80 & -384 \\ 7 & -119 & 420 \\ -3 & 39 & -120 \end{bmatrix}$$

The adjoint of matrix $[A]$ is $[C]^{\mathrm{T}}$,

$$adj(A) = [C]^{\mathrm{T}}$$
$$= \begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$

Hence

$$[A]^{-1} = \frac{1}{\det(A)} adj(A)$$
$$= \frac{1}{-84} \begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$
$$= \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

**If the inverse of a square matrix [A] exists, is it unique?**

Yes, the inverse of a square matrix is unique, if it exists. The proof is as follows. Assume that the inverse of $[A]$ is $[B]$ and if this inverse is not unique, then let another inverse of $[A]$ exist called $[C]$.

If $[B]$ is the inverse of $[A]$, then

$$[B][A]=[I]$$

Multiply both sides by $[C]$,

$$[B][A][C]=[I][C]$$

$$[B][A][C]=[C]$$

Since $[C]$ is inverse of $[A]$,

$$[A][C]=[I]$$

Multiply both sides by $[B]$,

$$[B][I]=[C]$$

$$[B]=[C]$$

This shows that $[B]$ and $[C]$ are the same. So the inverse of $[A]$ is unique.

**Key Terms:**

*Consistent system*
*Inconsistent system*
*Infinite solutions*
*Unique solution*
*Rank*
*Inverse*