Chapter 07.04
Romberg Rule of Integration

After reading this chapter, you should be able to:

1. *derive the Romberg rule of integration, and*
2. *use the Romberg rule of integration to solve problems.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Romberg rule of approximating integrals of the form

$$I = \int_a^b f(x)dx \tag{1}$$

where

$f(x)$ is called the integrand

$a =$ lower limit of integration

$b =$ upper limit of integration

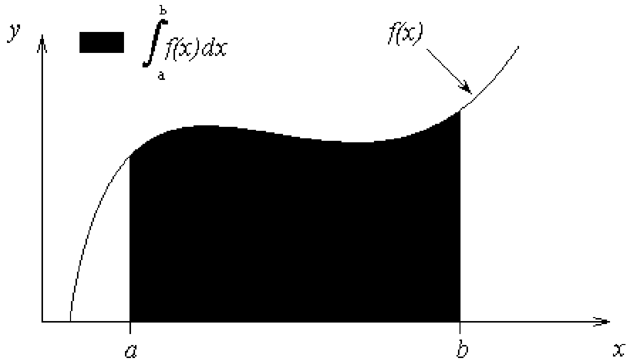

Figure 1 Integration of a function.

Error in Multiple-Segment Trapezoidal Rule

The true error obtained when using the multiple segment trapezoidal rule with n segments to approximate an integral

$$\int_a^b f(x)dx$$

is given by

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \qquad (2)$$

where for each i, ξ_i is a point somewhere in the domain $[a+(i-1)h, a+ih]$, and the term $\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ can be viewed as an approximate average value of $f''(x)$ in $[a,b]$. This leads us to say that the true error E_t in Equation (2) is approximately proportional to

$$E_t \approx \alpha\frac{1}{n^2} \qquad (3)$$

for the estimate of $\int_a^b f(x)dx$ using the n-segment trapezoidal rule.

Table 1 shows the results obtained for

$$\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

using the multiple-segment trapezoidal rule.

Table 1 Values obtained using multiple segment trapezoidal rule for
$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$.

n	Approximate Value	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
1	11868	-807	7.296	---
2	11266	-205	1.854	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

The true error for the 1-segment trapezoidal rule is -807, while for the 2-segment rule, the true error is -205. The true error of -205 is approximately a quarter of -807. The true error gets approximately quartered as the number of segments is doubled from 1 to 2. The same trend is observed when the number of segments is doubled from 2 to 4 (the true error for 2-segments is -205 and for four segments is -51.5). This follows Equation (3).
This information, although interesting, can also be used to get a better approximation of the integral. That is the basis of Richardson's extrapolation formula for integration by the trapezoidal rule.

Richardson's Extrapolation Formula for Trapezoidal Rule

The true error, E_t, in the n-segment trapezoidal rule is estimated as

$$E_t \approx \alpha \frac{1}{n^2}$$

$$E_t \approx \frac{C}{n^2} \tag{4}$$

where C is an approximate constant of proportionality.
Since

$$E_t = TV - I_n \tag{5}$$

where

TV = true value

I_n = approximate value using n-segments

Then from Equations (4) and (5),

$$\frac{C}{n^2} \approx TV - I_n \tag{6}$$

If the number of segments is doubled from n to $2n$ in the trapezoidal rule,

$$\frac{C}{(2n)^2} \approx TV - I_{2n} \tag{7}$$

Equations (6) and (7) can be solved simultaneously to get

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3} \tag{8}$$

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_0^{100} f(x)dx$$

Where,

$$f(x) = 0,\ 0 < x < 30$$
$$= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778,\ \ 30 \le x \le 172$$
$$= 0,\ \ 172 < x < 200$$

Table 2 Values obtained for Trapezoidal rule.

n	Trapezoidal Rule
1	−0.85000
2	63.493
4	36.062
8	55.753

a) Use Richardson's extrapolation formula to find the value of the integral. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) $I_2 = 63.493$

$I_4 = 36.061$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing $n = 2$,

$$TV \approx I_4 + \frac{I_4 - I_2}{3}$$
$$\approx 36.062 + \frac{36.062 - 63.493}{3}$$
$$\approx 26.917$$

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$I = \int_0^{100} f(x)dx$$
$$= 60.793$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 60.793 - 26.918$$
$$= 33.876$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{60.793 - 26.918}{60.793}\right| \times 100\,\%$$
$$= 55.724\,\%$$

Table 3 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Table 3 Values obtained using Richardson's extrapolation formula for Trapezoidal rule for example 1.

n	Trapezoidal Rule	$\|\in_t\|$ for Trapezoidal Rule %	Richardson's Extrapolation	$\|\in_t\|$ for Richardson's Extrapolation %
1	−0.85000	101.40	--	--
2	63.498	4.4494	84.947	39.733
4	36.062	40.681	26.917	55.724
8	55.754	8.2885	62.318	2.5092

Romberg Integration

Romberg integration is the same as Richardson's extrapolation formula as given by Equation (8). However, Romberg used a recursive algorithm for the extrapolation as follows.

The estimate of the true error in the trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

Since the segment width, h, is given by

$$h = \frac{b-a}{n}$$

Equation (2) can be written as

$$E_t = -\frac{h^2(b-a)}{12}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \tag{9}$$

The estimate of true error is given by

$$E_t \approx Ch^2 \tag{10}$$

It can be shown that the exact true error could be written as

$$E_t = A_1h^2 + A_2h^4 + A_3h^6 + \ldots \tag{11}$$

and for small h,

$$E_t = A_1h^2 + O(h^4) \tag{12}$$

Since we used $E_t \approx Ch^2$ in the formula (Equation (12)), the result obtained from Equation (10) has an error of $O(h^4)$ and can be written as

$$\begin{aligned}(I_{2n})_R &= I_{2n} + \frac{I_{2n} - I_n}{3} \\ &= I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}\end{aligned} \tag{13}$$

where the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by the sign =.

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3} \tag{14}$$

then

$$TV \approx (I_{4n})_R + C\left(\frac{h}{2}\right)^4$$

From Equation (13) and (14),

$$\begin{aligned}TV &\approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}\end{aligned} \tag{15}$$

The above equation now has the error of $O(h^6)$. The above procedure can be further improved by using the new values of the estimate of the true value that has the error of $O(h^6)$ to give an estimate of $O(h^8)$.

Based on this procedure, a general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1},\ k \geq 2 \tag{16}$$

The index k represents the order of extrapolation. For example, $k=1$ represents the values obtained from the regular trapezoidal rule, $k=2$ represents the values obtained using the true error estimate as $O(h^2)$, etc. The index j represents the more and less accurate estimate of the integral. The value of an integral with a $j+1$ index is more accurate than the value of the integral with a j index.

For $k=2$, $j=1$,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1}$$

$$= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

For $k=3$, $j=1$,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1}$$

$$= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \tag{17}$$

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_0^{100} f(x)dx$$

Where,

$$f(x) = 0,\ 0 < x < 30$$

$$= -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778,\ 30 \le x \le 172$$

$$= 0,\ 172 < x < 200$$

Use Romberg's rule to find the value of the integral. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = -0.85000$$

$$I_{1,2} = 63.498$$

$$I_{1,3} = 36.062$$

$$I_{1,4} = 55.754$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values,

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 63.498 + \frac{63.498 - (-0.85000)}{3} \\ &= 84.947 \end{aligned}$$

Similarly

$$\begin{aligned} I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 36.062 + \frac{36.062 - 63.498}{3} \\ &= 26.917 \end{aligned}$$

$$\begin{aligned} I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\ &= 55.754 + \frac{55.754 - 36.062}{3} \\ &= 62.318 \end{aligned}$$

For the second order extrapolation values,

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 26.917 + \frac{26.917 - 84.947}{15} \\ &= 23.048 \end{aligned}$$

Similarly

$$\begin{aligned} I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\ &= 62.318 + \frac{62.318 - 26.917}{15} \\ &= 64.678 \end{aligned}$$

For the third order extrapolation values,

$$\begin{aligned} I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\ &= 64.678 + \frac{64.678 - 23.048}{63} \\ &= 65.339 \end{aligned}$$

Table 2 shows these increased correct values in a tree graph.

Table 3 Improved estimates of value of integral using Romberg integration.

		1st Order	2nd Order	3rd Order
1-segment	−0.85000			
		84.947		
2-segment	63.498		23.048	
		26.917		65.339
4-segment	36.062		64.678	
		62.318		
8-segment	55.754			

INTEGRATION	
Topic	Romberg Rule
Summary	Textbook notes of Romberg Rule of integration.
Major	Computer Engineering
Authors	Autar Kaw
Date	November 14, 2012
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