Chapter 07.03

Simpson's 1/3 Rule of Integration

After reading this chapter, you should be able to

1. *derive the formula for Simpson's 1/3 rule of integration,*
2. *use Simpson's 1/3 rule it to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 1/3 rule of integration,*
4. *use multiple-segment Simpson's 1/3 rule of integration to solve integrals, and*
5. *derive the true error formula for multiple-segment Simpson's 1/3 rule.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

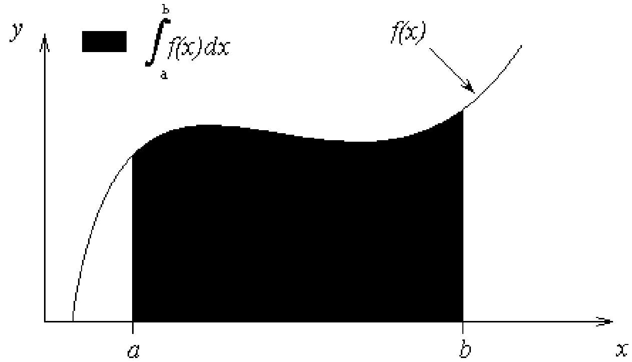

Figure 1 Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1 x + a_2 x^2 .$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson 1/3 rule, the interval $[a,b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

Method 2:

Simpson's 1/3 rule can also be derived by approximating $f(x)$ by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\dfrac{f(b) - f\left(\dfrac{a+b}{2}\right)}{b - \dfrac{a+b}{2}} - \dfrac{f\left(\dfrac{a+b}{2}\right) - f(a)}{\dfrac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\begin{aligned}
\int_a^b f(x)dx &\approx \int_a^b f_2(x)dx \\
&= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right) \right] dx \\
&= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right) \right]_a^b \\
&= b_0(b-a) + b_1\left(\frac{b^2-a^2}{2} - a(b-a)\right) \\
&\quad + b_2\left(\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)
\end{aligned}$$

Substituting values of b_0, b_1, and b_2 into this equation yields the same result as before

$$\begin{aligned}
\int_a^b f(x)dx &\approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\
&= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\end{aligned}$$

Method 3:

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \dfrac{a+b}{2}\right)(x-b)}{\left(a - \dfrac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\dfrac{a+b}{2} - a\right)\left(\dfrac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \dfrac{a+b}{2}\right)}{(b-a)\left(b - \dfrac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\int_a^b f_2(x)dx = \left[\frac{\frac{x^3}{3} - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b) \right]_a^b$$

$$= \frac{\frac{b^3-a^3}{3} - \frac{(a+3b)(b^2-a^2)}{4} + \frac{b(a+b)(b-a)}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a)$$

$$+ \frac{\frac{b^3-a^3}{3} - \frac{(a+b)(b^2-a^2)}{2} + ab(b-a)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right)$$

$$+ \frac{\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].$$

<u>Method 4:</u>

Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_a^b 1dx,\ \int_a^b xdx,$ and $\int_a^b x^2dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1dx = b - a = c_1 + c_2 + c_3$$

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

Solving the above three equations for c_0, c_1 and c_2 give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

The integral from the first method

$$\int_a^b f(x)\,dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2)\,dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_0^{100} f(x)\,dx$$

where

$$f(x) = 0, \ 0 < x < 30$$
$$= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778, \ \ 30 \le x \le 172$$
$$= 0, \ \ 172 < x < 200$$

a) Use Simpson's 1/3 Rule to find the probability.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) $$I \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$a = 0$$
$$b = 100$$
$$\frac{a+b}{2} = 50$$
$$f(x) = 0, \ 0 < x < 30$$
$$= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778, \ \ 30 \le x \le 172$$
$$= 0, \ \ 172 < x < 200$$

$$f(0) = 0$$
$$f(100) = -9.1688\times10^{-6}\times(100)^3 + 2.7961\times10^{-3}\times(100)^2 - 2.8487\times10^{-1}\times(100) + 9.6778$$
$$= -0.017000$$
$$f(50) = -9.1688\times10^{-6}\times(50)^3 + 2.7961\times10^{-3}\times(50)^2 - 2.8487\times10^{-1}\times(50) + 9.6778$$
$$= 1.2784$$
$$I \approx \left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$
$$\approx \left(\frac{100-0}{6}\right)[f(0)+4f(50)+f(100)]$$
$$\approx \left(\frac{100}{6}\right)[0+4(1.2784)+(-0.017)]$$
$$\approx 84.947$$

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$I = \int_0^{100} f(x)dx$$
$$= 60.793$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 60.793 - (84.947)$$
$$= -24.154$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{60.793 - (84.947)}{60.793}\right| \times 100\,\%$$
$$= 39.732\,\%$$

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$$x_0 = a$$
$$x_n = b$$
$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$
$$i = 2,\ 4,\ \ldots,\ n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + 2h\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$=\frac{h}{3}\left[f(x_0)+4\{f(x_1)+f(x_3)+...+f(x_{n-1})\}+2\{f(x_2)+f(x_4)+...+f(x_{n-2})\}+f(x_n)\right]$$

$$=\frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1}f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2}f(x_i)+f(x_n)\right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1}f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2}f(x_i)+f(x_n)\right]}$$

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I=\int_0^{100} f(x)dx$$

where

$$f(x)=0,\ 0<x<30$$

$$=-9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,\ 30\le x\le 172$$

$$=0,\ 172<x<200$$

a) Use four segment Simpson's 1/3 Rule to find the value of the integral
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) Using n segment Simpson's 1/3 Rule,

$$I\approx\frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1}f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2}f(x_i)+f(x_n)\right]$$

$n=4$

$a=0$

$b=100$

$$h=\frac{b-a}{n}$$

$$=\frac{100-0}{4}$$

$$= 25$$

$$\begin{aligned} f(x) &= 0,\ 0 < x < 30 \\ &= -9.1688\times 10^{-6}x^3 + 2.7961\times 10^{-3}x^2 - 2.8487\times 10^{-1}x + 9.6778,\ \ 30 \le x \le 172 \\ &= 0,\ 172 < x < 200 \end{aligned}$$

So

$$f(x_0) = f(0)$$
$$f(0) = 0$$

$$f(x_1) = f(0+25)$$
$$= f(25)$$
$$f(25) = 0$$

$$f(x_2) = f(25+25)$$
$$= f(50)$$
$$f(50) = -9.1688\times 10^{-6}\times(50)^3 + 2.7961\times 10^{-3}\times(50)^2 - 2.8487\times 10^{-1}\times(50) + 9.6778$$
$$= 1.2784$$

$$f(x_3) = f(50+25)$$
$$= f(75)$$
$$f(75) = -9.1688\times 10^{-6}\times(75)^3 + 2.7961\times 10^{-3}\times(75)^2 - 2.8487\times 10^{-1}\times(75) + 9.6778$$
$$= 0.17253$$

$$f(x_4) = f(x_n)$$
$$= f(100)$$
$$f(100) = -9.1688\times 10^{-6}\times(100)^3 + 2.7961\times 10^{-3}\times(100)^2 - 2.8487\times 10^{-1}\times(100) + 9.6778$$
$$= 0.017000$$

$$I \approx \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$\approx \frac{100-0}{3(4)}\left[f(0) + 4\sum_{\substack{i=1\\ i=odd}}^{3} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{2} f(x_i) + f(100)\right]$$

$$\approx \frac{100}{12}\left[f(0) + 4f(x_1) + 4f(x_3) + 2f(x_2) + f(100)\right]$$

$$\approx \frac{25}{3}\left[f(0) + 4f(25) + 4f(75) + 2f(50) + f(100)\right]$$

$$\approx \frac{25}{3}\left[0+4(0)+4(0.17253)+2(1.2784)+(-0.017000)\right]$$
$$\approx 26.917$$

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$I = \int_0^{100} f(x)dx$$
$$= 60.793$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 60.793 - (26.917)$$
$$= 33.873$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{60.793-(26.917)}{60.793}\right| \times 100\,\%$$
$$= 55.724\,\%$$

Table 1 Values of Simpson's 1/3 Rule for Example 2 with multiple segments.

n	Approximate Value	E_t	$\lvert\in_t\rvert$ %
2	84.947	−24.154	39.732
4	26.917	33.876	55.724
6	66.606	−5.8138	9.5633
8	62.318	−1.5252	2.5088
10	85.820	−25.023	41.169

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given[1] by

[1] The $f^{(4)}$ in the true error expression stands for the fourth derivative of the function *f(x)*.

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the n segments Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_1)$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_2)$$

$$\vdots$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_i)$$

$$\vdots$$

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i$$

$$= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x), a < x < b$. Hence

$$E_t = -\frac{(b-a)^5}{90n^4} \overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

INTEGRATION	
Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	Computer Engineering
Authors	Autar Kaw, Michael Keteltas
Date	November 11, 2012
Web Site	http://numericalmethods.eng.usf.edu