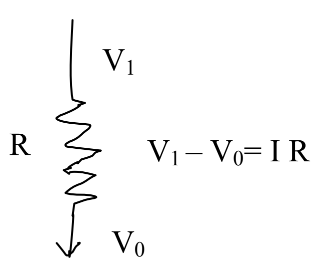
Chapter 08.00D

Physical Problem for Computer Engineering Ordinary Differential Equations

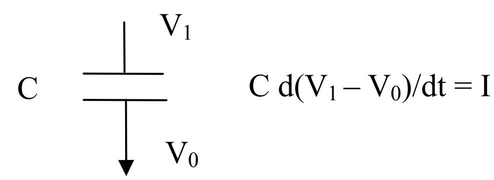
Problem Statement

Resistors and capacitors are fundamental elements of any circuit. Even the behavior of semiconductor devices in your computer can be modeled employing these basis elements (along with some others). A transistor is comprised of junctions of different kinds of materials, giving rise to interesting electrical properties. The electrical properties at these semiconductor junctions can be characterized using resistors (R) and capacitors (C), giving rise to the name "RC-model". In this module we will consider the electrical behavior of the simplest configuration of the RC elements and see how it can characterized using ordinary differential elements.

Two quantities that are important in electrical circuits are voltage (denoted here by V) and current (denoted here by I). The current through a resistor has a linear relationship with voltage and is defined by $V = IR$, where R, is constant used to quantify the resistor. This is called the Ohms law and you should have seen it one of your physics classes. A capacitor is a bit stranger element for which the rate of change of voltage across it is proportional to the current. Or in other words: $C\frac{dV}{dt} = I$, where C is called the capacitance. Voltage changes slowly across high capacitance. These elements are represented by diagrams as shown below.



Resistor:
$V_1 - V_0$ = Voltage drop
I = Current



Capacitor:
$V_1 - V_0$ = Voltage drop
I = Current

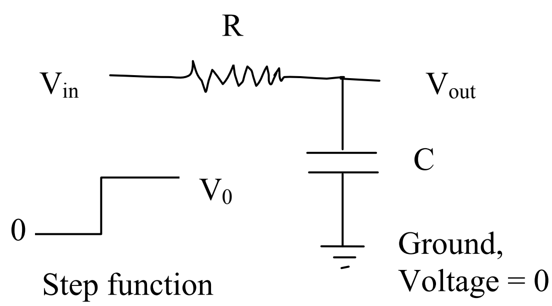


Figure 1. A simple RC circuit. As the input voltage is switched on at t=0, what happens to the output voltage.

Consider the simple combination of resistor and capacitor, as shown above. This kind of configuration is not rare and occurs in all power supplies to your computer. There are other components in addition to this, but this is one of the basic arrangements. The voltage at one end of the resistor is denoted by Vin and the voltage at the other end is denoted by Vout. The latter end is also connected to the capacitor, whose other end of which is connected to ground, which can be taken to be at zero voltage. The question we ask is what happens to the output voltage Vout when the input voltage Vin is switched on suddenly?

The voltage and current relationship across the resistor and the capacitor can be expressed, respectively as

$$V_{in} - V_{out} = IR \qquad \text{and} \qquad C\frac{dV_{out}}{dt} = I$$

where I is the current through the circuit. Eliminating this quantity from the above two equations, we have

$$V_{in} - V_{out} = RC\frac{dV_{out}}{dt}$$

For any input voltage profile we can solve this equation to arrive at the output voltage profile.

Worked Out Example

The sudden change in the input voltage can be modeled using the following function

$$V_{in} = \begin{cases} V_0 & \forall t \geq 0 \\ 0 & \forall t < 0 \end{cases}$$

This is also called a "step" function. For this input the differential equation modeling the output voltage for t > 0 is given by

$$RC\frac{dV_{out}}{dt} = V_0 - V_{out}$$

The solution of this equation we need some boundary conditions. We know the voltage at $t = 0$ is zero, i.e. $V_{out}(0) = 0$, and the voltage after a long time should be equal to the new input voltage, i.e $V_{out}(\infty) = V_0$. Using boundary conditions, in along with general solution form: $V_{out}(t) = c_o \exp(at) + c_1 \exp(-at) + c_2$, we can arrive at the following solution for the output voltage

$$V_{out}(t) = V_o(1 - \exp(-\frac{t}{RC}))$$

In the Figure 2 we see the plot of the transient output voltage for various values of the product RC. The output starts at zero and gradually converges to V0 = 1. The rate of convergence is dependent on the product RC.

QUESTIONS

1. If the for $V_{in} = x_1(t)$, the output voltage is given by the function $y_1(t)$, and for $V_{in} = x_2(t)$, the output voltage is given by the function $y_2(t)$, show that the output for the input $V_{in} = x_1(t) + x_2(t)$, is given by $y_1(t) + y_2(t)$.
2. If the for $V_{in} = x_1(t)$, the output voltage is given by the function $y_1(t)$, show that the output for the input $V_{in} = ax_1(t)$, is given by $ay_1(t)$.
3. Consider a pulse shaped input that is defined by:

$$V_{in}(t) = \begin{cases} 0 & \forall t < 0 \\ V_0 & 0 \le t \le T \\ 0 & t > T \end{cases}$$

   What is the form of the output?
   Hint: Express this input of in the form $x_1(t) - x_2(t)$ and then use the properties you have derived so far to arrive at the solution.
4. What can you say about the output as the ratio $\frac{T}{RC}$ is varied?
5. What can say about the behavior of this circuit when there is a voltage “spike” at the input?

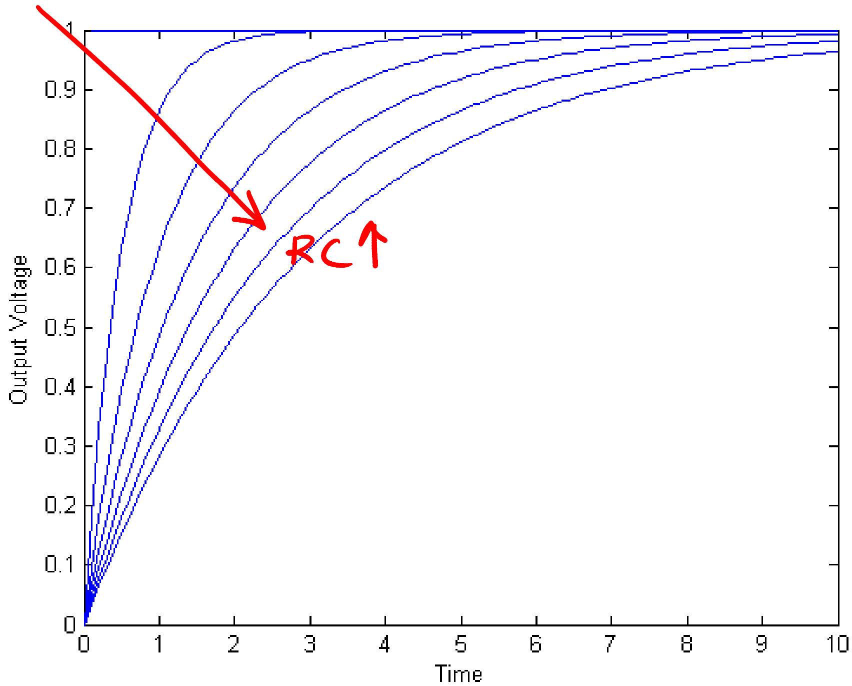


Figure 2. Output voltage, Vout(t), for various values of the product RC.

ORDINARY DIFFERENTIAL EQUATIONS	
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