Sources of Error

Major: All Engineering Majors
Authors: Autar Kaw, Luke Snyder

http://numericalmethods.eng.usf.edu
Transforming Numerical Methods Education for STEM
Undergraduates

Sources of Error

http://numericalmethods.eng.usf.edu

Two sources of numerical error

1) Round off error
2) Truncation error

Round-off Error

http://numericalmethods.eng.usf.edu

Round off Error

- Caused by representing a number approximately

$$\frac{1}{3} \cong 0.333333$$

$$\sqrt{2} \cong 1.4142...$$

Problems created by round off error

- 28 Americans were killed on February 25, 1991 by an Iraqi Scud missile in Dhahran, Saudi Arabia.

- The patriot defense system failed to track and intercept the Scud. Why?

Problem with Patriot missile

- Clock cycle of 1/10 seconds was represented in 24-bit fixed point register created an error of 9.5 x 10^{-8} seconds.

- The battery was on for 100 consecutive hours, thus causing an inaccuracy of

$$= 9.5 \times 10^{-8} \frac{\text{s}}{0.1\text{s}} \times 100\text{hr} \times \frac{3600\text{s}}{1\text{hr}}$$
$$= 0.342s$$

Problem (cont.)

- The shift calculated in the ranging system of the missile was 687 meters.

- The target was considered to be out of range at a distance greater than 137 meters.

Effect of Carrying Significant Digits in Calculations

http://numericalmethods.eng.usf.edu
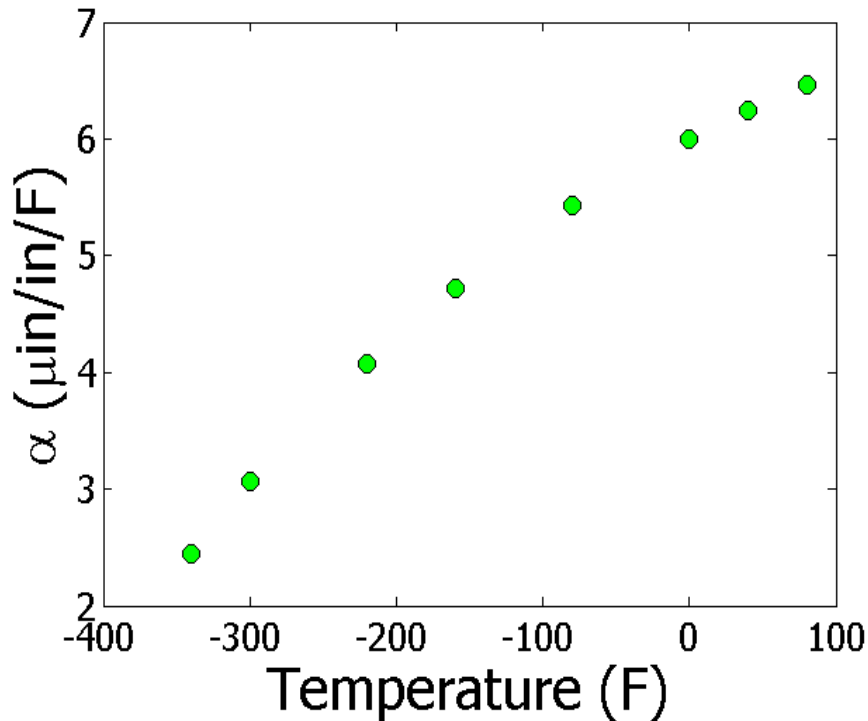
Find the contraction in the diameter

$$\Delta D = D \int_{T_a}^{T_c} \alpha(T) dT$$

$T_a = 80°F; \ T_c = -108°F; \ D = 12.363″$

$$\alpha = a_0 + a_1 T + a_2 T^2$$

Thermal Expansion Coefficient vs Temperature

T (°F)	α (μin/in/°F)
-340	2.45
-300	3.07
-220	4.08
-160	4.72
-80	5.43
0	6.00
40	6.24
80	6.47

Regressing Data in Excel (general format)

$$\alpha = -1E\text{-}05T^2 + 0.0062T + 6.0234$$

Observed and Predicted Values

$$\alpha = -1\text{E-}05\text{T}^2 + 0.0062\text{T} + 6.0234$$

T(^{o}F)	α (μin/in/^{o}F) Given	α (μin/in/^{o}F) Predicted
-340	2.45	2.76
-300	3.07	3.26
-220	4.08	4.18
-160	4.72	4.78
-80	5.43	5.46
0	6.00	6.02
40	6.24	6.26
80	6.47	6.46

Regressing Data in Excel (scientific format)

$$\alpha = -1.2360\text{E-}05\,T^2 + 6.2714\text{E-}03\,T + 6.0234$$

Observed and Predicted Values

$$\alpha = -1.2360\text{E-}05\text{T}^2 + 6.2714\text{E-}03\text{T} + 6.0234$$

T(°F)	α (μin/in/°F) Given	α (μin/in/°F) Predicted
-340	2.45	2.46
-300	3.07	3.03
-220	4.08	4.05
-160	4.72	4.70
-80	5.43	5.44
0	6.00	6.02
40	6.24	6.25
80	6.47	6.45

Observed and Predicted Values

$$\alpha = -1.2360\text{E-}05\text{T}^2 + 6.2714\text{E-}03\text{T} + 6.0234$$

$$\alpha = -1\text{E-}05\text{T}^2 + 0.0062\text{T} + 6.0234$$

$T(^oF)$	α (μin/in/^oF) Given	α (μin/in/^oF) Predicted	α (μin/in/^oF) Predicted
-340	2.45	2.46	2.76
-300	3.07	3.03	3.26
-220	4.08	4.05	4.18
-160	4.72	4.70	4.78
-80	5.43	5.44	5.46
0	6.00	6.02	6.02
40	6.24	6.25	6.26
80	6.47	6.45	6.46

THE END

Truncation Error

http://numericalmethods.eng.usf.edu

Truncation error

- Error caused by truncating or approximating a mathematical procedure.

Example of Truncation Error

Taking only a few terms of a Maclaurin series to approximate e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

If only 3 terms are used,

$$Truncation \ Error = e^x - \left(1 + x + \frac{x^2}{2!} \right)$$

Another Example of Truncation Error

Using a finite Δx to approximate $f'(x)$

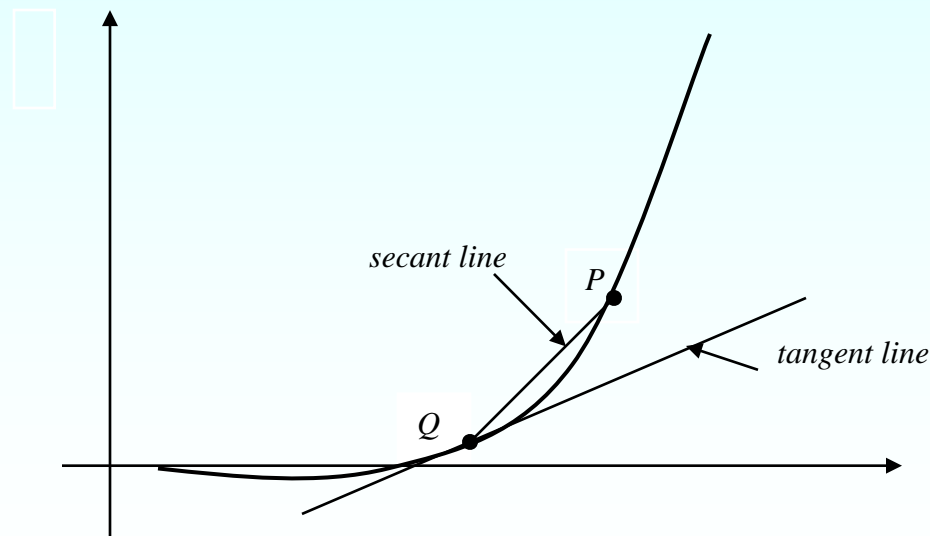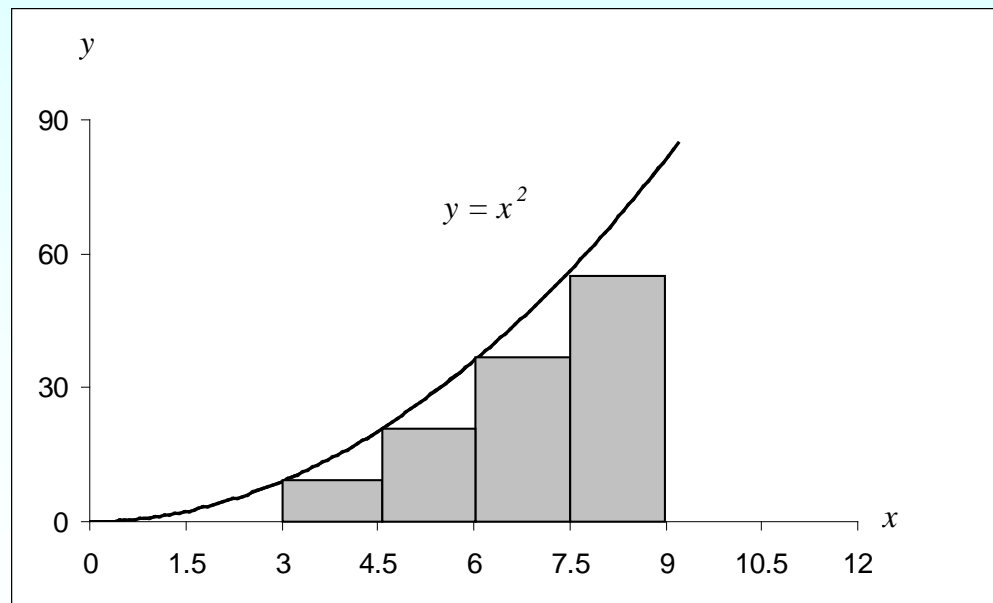$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Figure 1. Approximate derivative using finite Δx

Another Example of Truncation Error

Using finite rectangles to approximate an integral.

Example 1 —Maclaurin series

Calculate the value of $e^{1.2}$ with an absolute relative approximate error of less than 1%.

$$e^{1.2} = 1 + 1.2 + \frac{1.2^2}{2!} + \frac{1.2^3}{3!} + ...$$

n	$e^{1.2}$	E_a	$\left| \in_a \right| \%$
1	1	__	___
2	2.2	1.2	54.545
3	2.92	0.72	24.658
4	3.208	0.288	8.9776
5	3.2944	0.0864	2.6226
6	3.3151	0.020736	0.62550

6 terms are required. How many are required to get at least 1 significant digit correct in your answer?

Example 2 —Differentiation

Find $f'(3)$ for $f(x) = x^2$ using $f'(x) \approx \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$
and $\Delta x = 0.2$

$$f'(3) = \frac{f(3 + 0.2) - f(3)}{0.2}$$

$$= \frac{f(3.2) - f(3)}{0.2} = \frac{3.2^2 - 3^2}{0.2} = \frac{10.24 - 9}{0.2} = \frac{1.24}{0.2} = 6.2$$
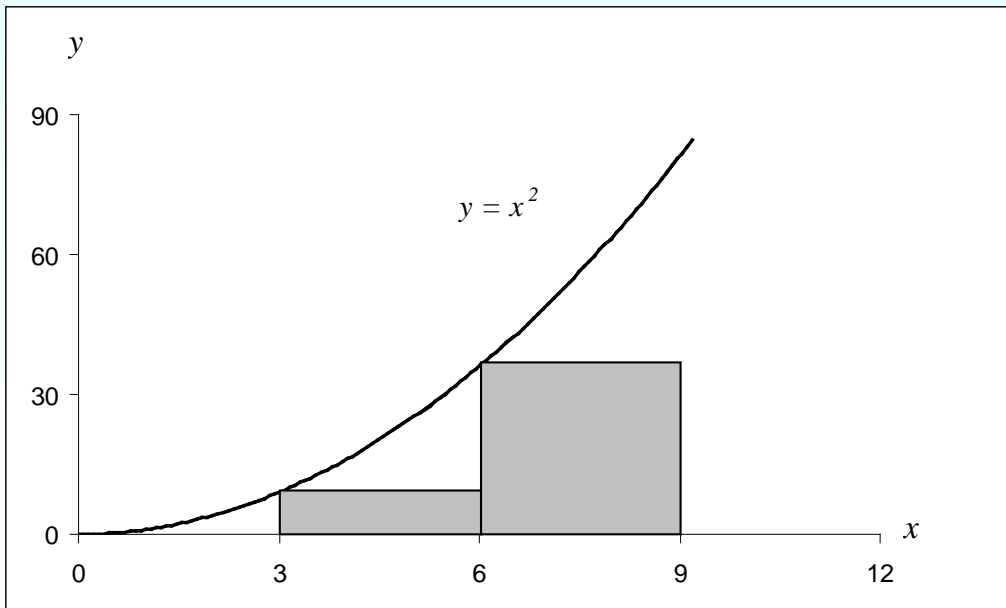
The actual value is

$$f'(x) = 2x, \quad f'(3) = 2 \times 3 = 6$$

Truncation error is then, $6 - 6.2 = -0.2$

Can you find the truncation error with $\Delta x = 0.1$ 24

Example 3 — Integration

Use two rectangles of equal width to approximate the area under the curve for $f(x) = x^2$ over the interval $[3,9]$

$$\int_3^9 x^2 \, dx$$

Integration example (cont.)

Choosing a width of 3, we have

$$\int_3^9 x^2 dx = (x^2)\Big|_{x=3}(6-3) + (x^2)\Big|_{x=6}(9-6)$$

$$= (3^2)3 + (6^2)3$$

$$= 27 + 108 = 135$$

Actual value is given by

$$\int_3^9 x^2 dx = \left[\frac{x^3}{3}\right]_3^9 = \left[\frac{9^3 - 3^3}{3}\right] = 234$$

Truncation error is then

$$234 - 135 = 99$$

Can you find the truncation error with 4 rectangles?

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/sources_of_error.html

THE END