Differentiation-Discrete Functions

Electric Engineering Majors

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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite 'Δx'

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

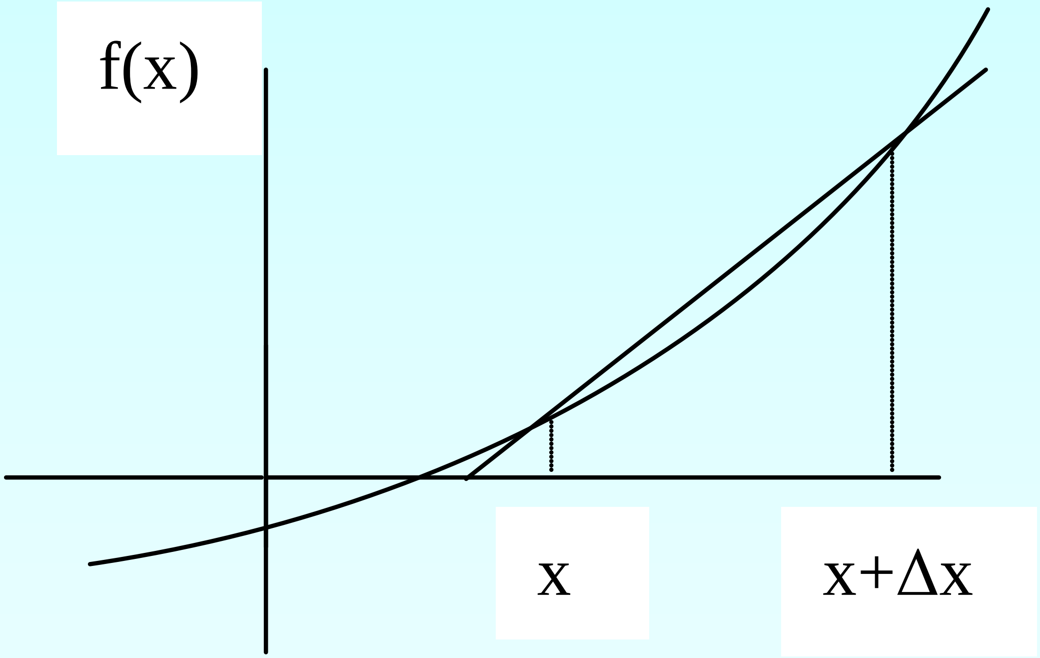

Figure 1 Graphical Representation of forward difference approximation of first derivative.

Example 1

To increase the reliability and life of a switch, one needs to turn the switch off as close to zero crossing as possible. To find this time of zero crossing, the value of $E(t)/E'(t)$ is to be found at all times given in Table 1, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at $t=10$.

See Table 1 for voltage as a function of time data.

Use Forward Divided Difference approximation of the first derivative to calculate $E(t)/E'(t)$ at $t=10$. Use a step size of $\Delta t=1$.

Example 1 Cont.

Table 1 Voltage as a function of time.

Time, $t\ (\text{s})$	**Voltage,** $E(t)\ (\text{V})$		**Time,** $t\ (\text{s})$	**Voltage,** $E(t)\ (\text{V})$
1	0.62161		13	−0.21078
2	0.36236		14	0.087499
3	0.070737		15	0.37798
4	−0.22720		16	0.63469
5	−0.50485		17	0.83471
6	−0.73739		18	0.96017
7	−0.90407		19	0.99986
8	−0.98999		20	0.95023
9	−0.98748		21	0.81573
10	−0.89676		22	0.60835
11	−0.72593		23	0.34664
12	−0.49026		24	0.053955

Example 1 Cont.

Solution

$$E'(t_i) \approx \frac{E(t_{i+1}) - E(t_i)}{\Delta t}$$

$$\begin{aligned} t_i &= 10 \\ t_{i+1} &= 11 \\ \Delta t &= t_{i+1} - t_i \\ &= 11 - 10 \\ &= 1 \end{aligned}$$

$$\begin{aligned} E'(10) &\approx \frac{E(11) - E(10)}{\Delta t} \\ &\approx \frac{-0.72593 - (-0.89676)}{1} \\ &\approx 0.17083 \text{ V/s} \end{aligned}$$

Example 1 Cont.

$$\frac{E(10)}{E^{'}(10)} \approx \frac{-0.89676}{0.17083}$$

$$\approx -5.2495\,\text{s}$$

Direct Fit Polynomials

In this method, given $'n+1'$ data points $(x_0, y_0),(x_1, y_1),(x_2, y_2),\ldots,(x_n, y_n)$ one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1}x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1)a_{n-1}x^{n-2} + na_n x^{n-1}$$

Similarly other derivatives can be found.

Example 2-Direct Fit Polynomials

To increase the reliability and life of a switch, one needs to turn the switch off as close to zero crossing as possible. To find this time of zero crossing, the value of $E(t)/E'(t)$ is to be found at all times given in Table 2, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at $t=10$.

See Table 2 for voltage as a function of time data.

Using the third order polynomial interpolant for Voltage, find the value of $E(t)/E'(t)$ at $t=10$.

Example 2-Direct Fit Polynomials cont.

Table 2 Voltage as a function of time.

Time, t (s)	**Voltage,** $E(t)$ (V)		**Time,** t (s)	**Voltage,** $E(t)$ (V)
1	0.62161		13	−0.21078
2	0.36236		14	0.087499
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Example 2-Direct Fit Polynomials cont.

Solution

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$E(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Since we want to find the voltage at $t = 10$, and we are using a third order polynomial, we need to choose the four points closest to $t = 10$ and that also bracket $t = 10$ to evaluate it.

The four points are $t_0 = 8$, $t_1 = 9$, $t_2 = 10$ and $t_3 = 11$.

$$t_o = 8, \quad E(t_o) = -0.98999$$

$$t_1 = 9, \quad E(t_1) = -0.98748$$

$$t_2 = 10, \quad E(t_2) = -0.89676$$

$$t_3 = 11, \quad E(t_3) = -0.72593$$

Example 2-Direct Fit Polynomials cont.

such that

$$E(8) = -0.98999 = a_0 + a_1(8) + a_2(8)^2 + a_3(8)^3$$

$$E(9) = -0.98748 = a_0 + a_1(9) + a_2(9)^2 + a_3(9)^3$$

$$E(10) = -0.89676 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$E(11) = -0.72593 = a_0 + a_1(11) + a_2(11)^2 + a_3(11)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.98999 \\ -0.98748 \\ -0.89676 \\ -0.72593 \end{bmatrix}$$

Example 2-Direct Fit Polynomials cont.

Solving the above four equations gives

$$a_0 = 3.1382$$

$$a_1 = -1.0742$$

$$a_2 = 0.080582$$

$$a_3 = -0.0013510$$

Hence

$$E(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$= 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3, \quad 8 \le t \le 11$$

Example 2-Direct Fit Polynomials cont.

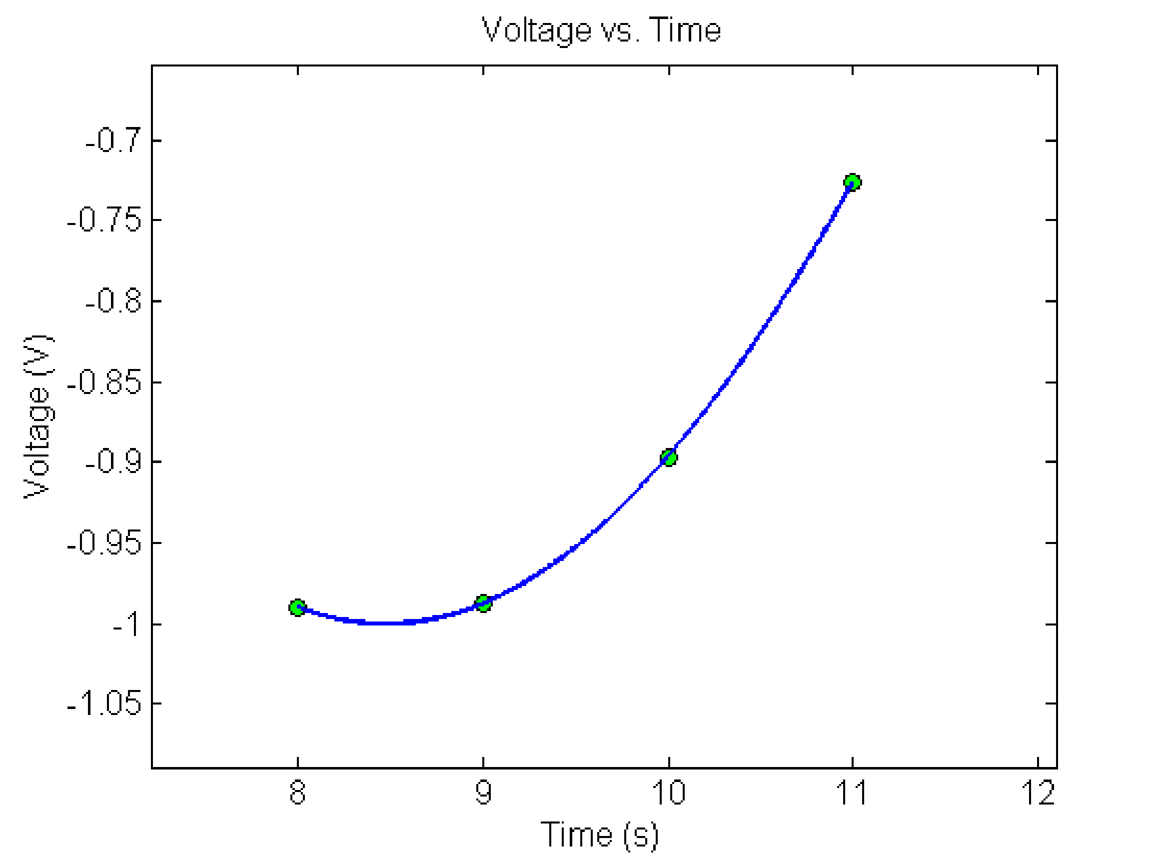

Figure 2 Graph of voltage of the switch vs. time.

Example 2-Direct Fit Polynomials cont.

The derivative of voltage at t=10 is given by

$$E^{'}(10) = \frac{d}{dt}E(t)\Big|_{t=10}$$

Given that

,

$$E(t) = 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3,\ 8 \le t \le 11$$

$$\begin{aligned} E^{'}(t) &= \frac{d}{dt}E(t) \\ &= \frac{d}{dt}\left(3.13812 - 1.0742t + 0.080582t^2 - 0.0013510t^3\right) \\ &= -1.0742 + 0.16116t - 0.0040530t^2, \quad 8 \le t \le 11 \end{aligned}$$

Example 2-Direct Fit Polynomials cont.

$$E'(10) = -1.0742 + 0.16116t - 0.0040530t^2$$
$$= 0.13210 \text{ V/s}$$

$$\frac{E(10)}{E'(10)} = \frac{-0.89676}{0.13210}$$
$$= -6.7872 \text{ s}$$

Lagrange Polynomial

In this method, given $(x_1, y_1), \ldots, (x_n, y_n)$, one can fit a $(n-1)^{th}$ order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where ' n ' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Lagrange Polynomial Cont.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating equation (2) gives

Lagrange Polynomial Cont.

$$f_2'(x) = \frac{2x-(x_1+x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-(x_0+x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2x-(x_0+x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{2}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Example 3

To increase the reliability and life of a switch, one needs to turn the switch off as close to zero crossing as possible. To find this time of zero crossing, the value of $E(t)/E'(t)$ is to be found at all times given in Table 3, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at $t = 10$.

See Table 3 for voltage as a function of time data.

Use the second order Lagrangian polynomial interpolation to calculate the value of $E(t)/E'(t)$ at $t = 10$.

Example 3 Cont.

Table 2 Voltage as a function of time.

Time, t (s)	**Voltage,** $E(t)$ (V)		**Time,** t (s)	**Voltage,** $E(t)$ (V)
1	0.62161		13	−0.21078
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12	−0.49026		24	0.053955

Example 3 Cont.

Solution:

For second order Lagrangian polynomial interpolation, we choose the voltage given by

$$E(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)E(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)E(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)E(t_2)$$

Since we want to find the voltage at $t = 10$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t = 10$ that also bracket $t = 10$ to evaluate it.

The three points are $t_0 = 9$, $t_1 = 10$, and $t_2 = 11$.

Differentiating the above equation gives

$$E'(t) = \frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}E(t_0) + \frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}E(t_1) + \frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}E(t_2)$$

Example 3 Cont.

Hence

$$E'(10) = \frac{2(10)-(10+11)}{(9-10)(9-11)}(-0.98748) + \frac{2(10)-(9+11)}{(10-9)(10-11)}(-0.89676) + \frac{2(10)-(9+10)}{(11-9)(11-10)}(-0.72593)$$

$$= -0.5(-0.98748) + 0(-0.89676) + 0.5(-0.72593)$$

$$= 0.13077 \text{ V/s}$$

$$\frac{E(10)}{E'(10)} = \frac{-0.89676}{0.13077}$$

$$= -6.8573\,\text{s}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/discrete_02dif.html

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