Chapter 02.03
Differentiation of Discrete Functions

After reading this chapter, you should be able to:

1. *find approximate values of the first derivative of functions that are given at discrete data points, and*
2. *use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

Forward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For a finite Δx,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

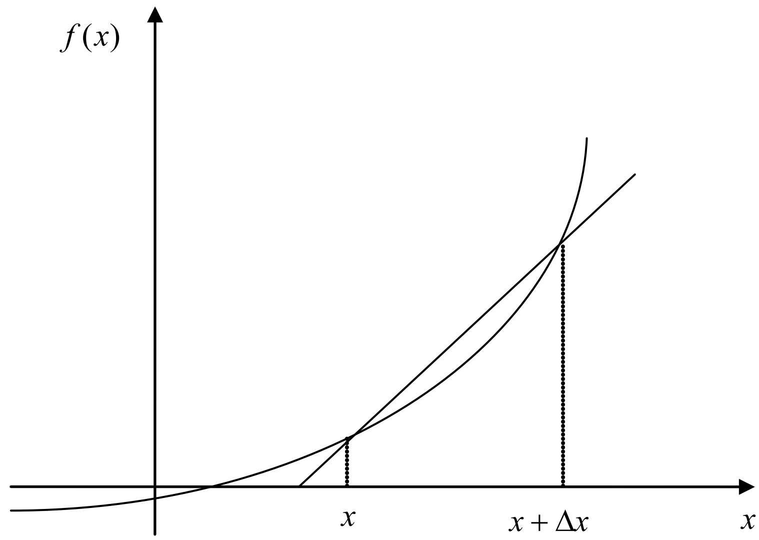

Figure 1 Graphical representation of forward difference approximation of first derivative.

So given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the value of $f'(x)$ for $x_i \le x \le x_{i+1}$, $i = 0, \ldots, n-1$, is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Example 1

To increase the reliability and life of a switch, one needs to turn the switch off as close to the zero-crossing as possible. To find this time of zero-crossing, the value of $E(t)/E'(t)$ is to be found at all times given in Table 1, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at $t = 10$.

Table 1 Voltage as a function of time.

Time, t (s)	Voltage, $E(t)$ (V)		Time, t (s)	Voltage, $E(t)$ (V)
1	0.62161		13	–0.210796
2	0.362358		14	0.087499
3	0.070737		15	0.377978
4	–0.227202		16	0.634693
5	–0.504846		17	0.834713
6	–0.737394		18	0.96017
7	–0.904072		19	0.999859
8	–0.989992		20	0.950233
9	–0.98748		21	0.815725
10	–0.896758		22	0.608351
11	–0.725932		23	0.346635
12	–0.490261		24	0.053955

Use the forward divided difference approximation of the first derivative to calculate $E(t)/E'(t)$ at $t = 10$. Use a step size of $\Delta t = 1$.

Solution

$$E'(t_i) \approx \frac{E(t_{i+1}) - E(t_i)}{\Delta t}$$

$$t_i = 10$$

$$t_{i+1} = 11$$

$$\begin{aligned}\Delta t &= t_{i+1} - t_i \\ &= 11 - 10 \\ &= 1\end{aligned}$$

$$\begin{aligned}E'(10) &\approx \frac{E(11) - E(10)}{1} \\ &= \frac{-0.725932 - (-0.896758)}{1} \\ &= 0.170826 \text{ V/s}\end{aligned}$$

$$\begin{aligned}\frac{E(10)}{E'(10)} &= \frac{-0.896758}{0.170826} \\ &= -5.2495 \text{ s}\end{aligned}$$

Direct Fit Polynomials

In this method, given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1} + a_n x^n$$

To find the first derivative,

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots\ldots + (n-1) a_{n-1} x^{n-2} + n a_n x^{n-1}$$

Similarly, other derivatives can also be found.

Example 2

To increase the reliability and life of a switch, one needs to turn the switch off as close to the zero-crossing as possible. To find this time of zero-crossing, the value of $E(t)/E'(t)$ is to be found at all times given in Table 2, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at $t = 10$.

Table 2 Voltage as a function of time.

Time, t (s)	Voltage, $E(t)$ (V)		Time, t (s)	Voltage, $E(t)$ (V)
1	0.62161		13	–0.210796
2	0.362358		14	0.087499
3	0.070737		15	0.377978
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12	–0.490261		24	0.053955

Using a third order polynomial interpolant for Voltage, find the value of $E(t)/E'(t)$ at $t=10$.

Solution

For a third order polynomial interpolation (also called cubic interpolation), we choose the voltage given by

$$E(t)=a_0+a_1t+a_2t^2+a_3t^3$$

Since we want to find the voltage at $t=10$, and we are using a third order polynomial, we need to choose the four points closest to $t=10$ that also bracket $t=10$ to evaluate it.

The four points are $t_0=8$, $t_1=9$, $t_2=10$ and $t_3=11$.

$$t_0=8,\ E(t_0)=-0.989992$$
$$t_1=9,\ E(t_1)=-0.98748$$
$$t_2=10,\ E(t_2)=-0.896758$$
$$t_3=11,\ E(t_3)=-0.725932$$

such that

$$E(8)=-0.989992=a_0+a_1(8)+a_2(8)^2+a_3(8)^3$$
$$E(9)=-0.98748=a_0+a_1(9)+a_2(9)^2+a_3(9)^3$$
$$E(10)=-0.896758=a_0+a_1(10)+a_2(10)^2+a_3(10)^3$$
$$E(11)=-0.725932=a_0+a_1(11)+a_2(11)^2+a_3(11)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.989992 \\ -0.98748 \\ -0.896758 \\ -0.725932 \end{bmatrix}$$

Solving the above gives

$$a_0 = 3.1382$$
$$a_1 = -1.0742$$
$$a_2 = 0.080582$$
$$a_3 = -0.001351$$

Hence

$$E(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$= 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3, \quad 8 \le t \le 11$$

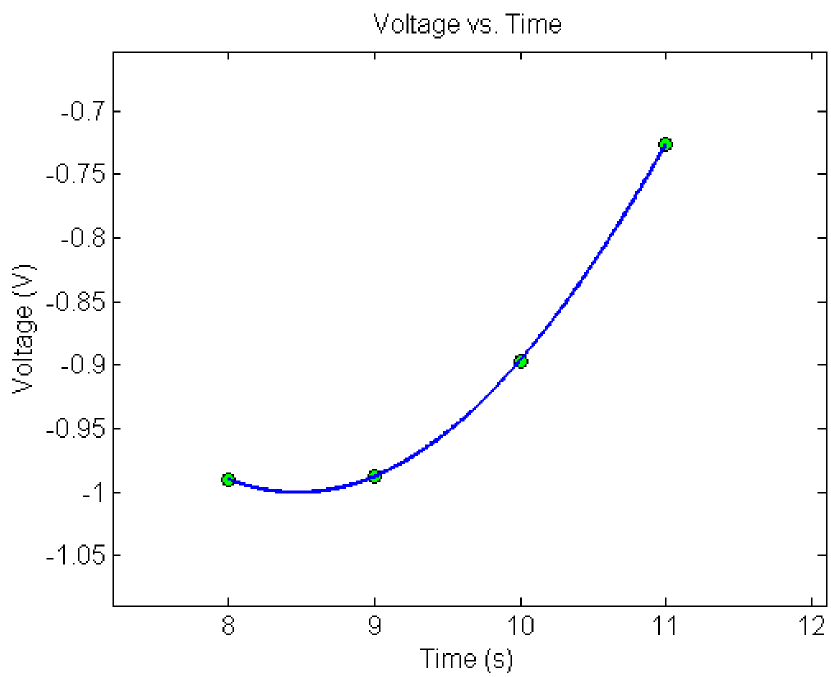

Figure 2 Graph of voltage of the switch vs. time.

The derivative of voltage at $t = 10$ is given by

$$E'(10) = \frac{d}{dt} E(t)\Big|_{t=10}$$

Given that $E(t) = 3.1382 - 1.0742t + 0.080582t^2 - 0.0013510t^3$, $8 \le t \le 11$,

$$E'(t) = \frac{d}{dt} E(t)$$
$$= \frac{d}{dt}\left(3.13812 - 1.0742t + 0.080582t^2 - 0.0013510t^3\right)$$
$$= -1.0742 + 0.16116t - 0.004053t^2, \; 8 \le t \le 11$$

$$E'(10) = -1.0742 + 0.16116(10) - 0.004053(10)^2$$
$$= 0.13210 \text{ V}/s$$
$$\frac{E(10)}{E'(10)} = \frac{-0.896758}{0.13210}$$
$$= -6.7872 \text{ s}$$

Lagrange Polynomial

In this method, given $(x_0, y_0), \ldots, (x_n, y_n)$, one can fit a n^{th} order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted.
Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.
For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2) is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Example 3

To increase the reliability and life of a switch, one needs to turn the switch off as close to the zero-crossing as possible. To find this time of zero-crossing, the value of $E(t)/E'(t)$ is to be found at all times given in Table 3, where $E(t)$ is the voltage and t is the time. To keep the problem simple, you are asked to find the approximate value of $E(t)/E'(t)$ at $t=10$.

Table 3 Voltage as a function of time.

Time, t (s)	Voltage, $E(t)$ (V)		Time, t (s)	Voltage, $E(t)$ (V)
1	0.62161		13	–0.210796
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12	–0.490261		24	0.053955

Use second order Lagrangian polynomial interpolation to calculate $E(t)/E'(t)$ at $t=10$.

Solution

For second order Lagrangian polynomial interpolation, we choose the voltage given by

$$E(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)E(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)E(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)E(t_2)$$

Since we want to find the voltage at $t=10$, and we are using a second order Lagrangian polynomial, we need to choose the three points closest to $t=10$ that also bracket $t=10$ to evaluate it. The three points are $t_0=9$, $t_1=10$, and $t_2=11$.

Differentiating the above equation gives

$$E'(t)=\frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)}E(t_0)+\frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)}E(t_1)+\frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)}E(t_2)$$

Hence

$$\begin{aligned}E'(10)&=\frac{2(10)-(10+11)}{(9-10)(9-11)}(-0.98748)+\frac{2(10)-(9+11)}{(10-9)(10-11)}(-0.896758)\\&\quad+\frac{2(10)-(9+10)}{(11-9)(11-10)}(-0.725932)\\&=-0.5(-0.98748)+0(-0.896758)+0.5(-0.725932)\\&=0.13077\ \text{V/s}\end{aligned}$$

$$\frac{E(10)}{E'(10)} = \frac{-0.896758}{0.13077}$$
$$= -6.8573 \text{ s}$$

DIFFERENTIATION	
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