Bisection Method

Electrical Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

Bisection Method

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Basis of Bisection Method

Theorem An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\ f(x_u) < 0$.

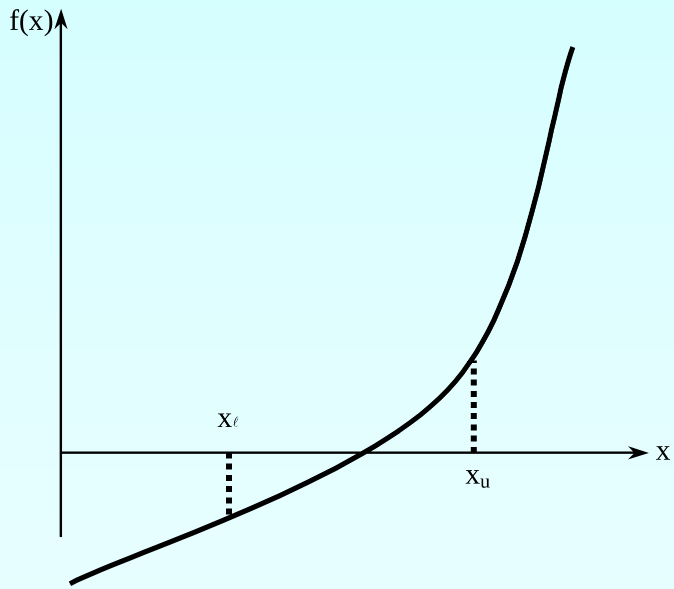


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

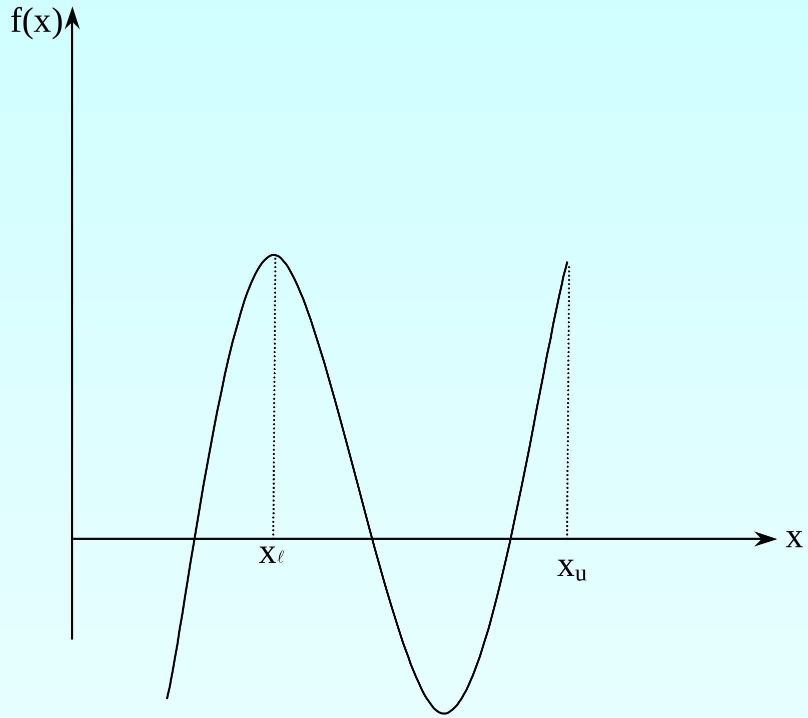


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method

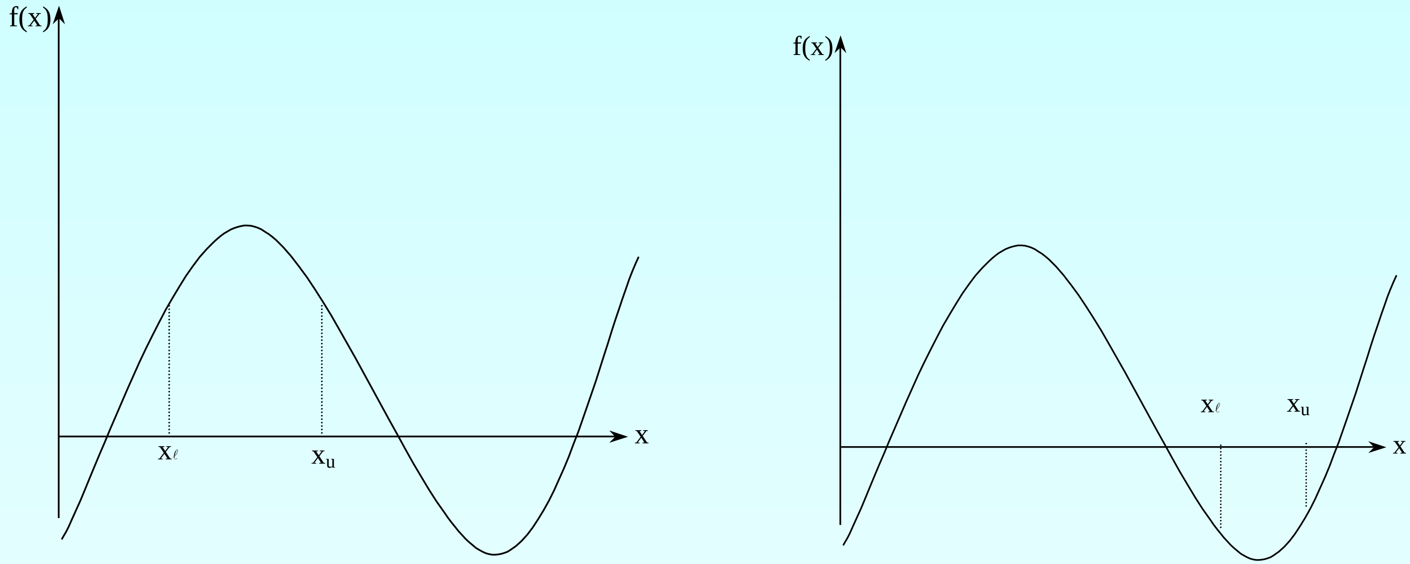


Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

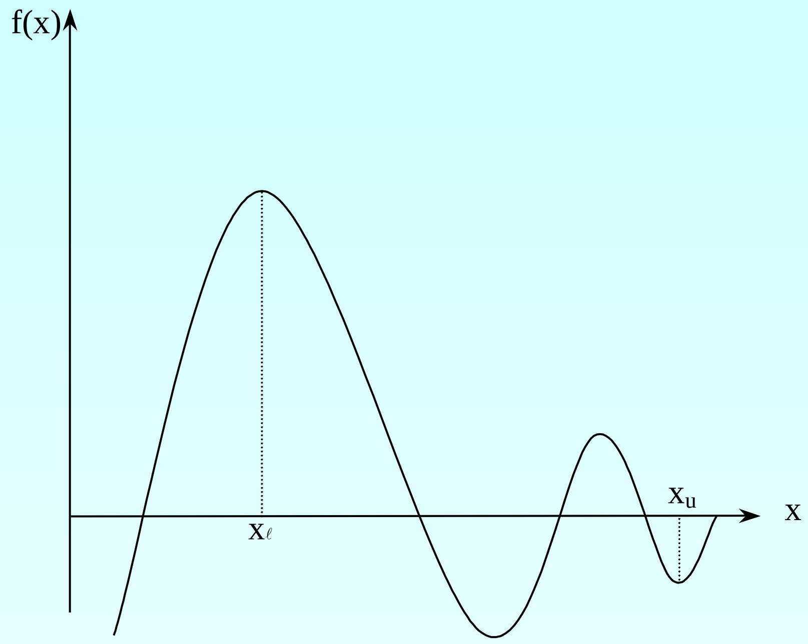


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u. This was demonstrated in Figure 1.

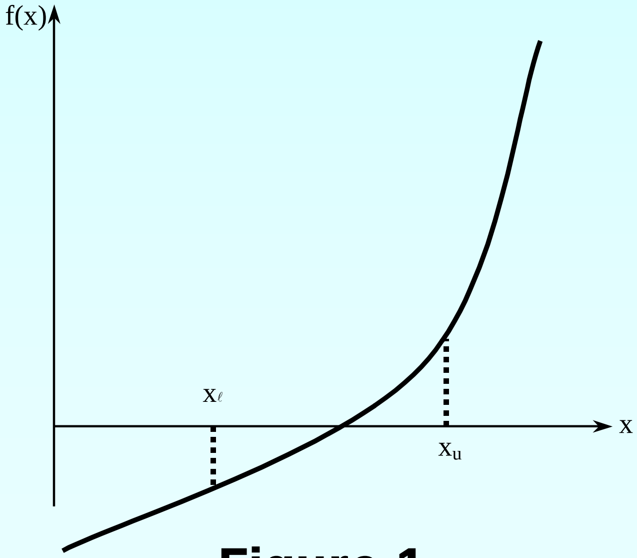


Figure 1

Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

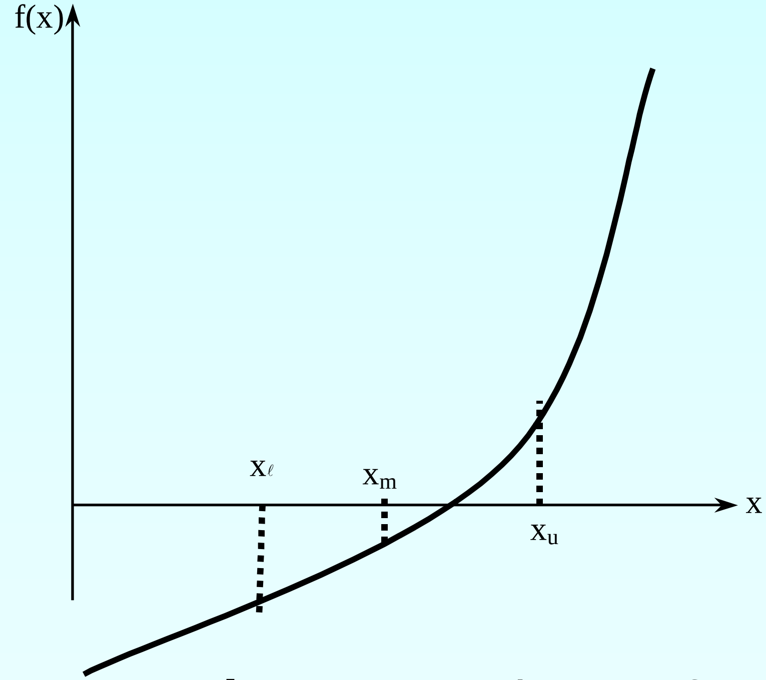


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

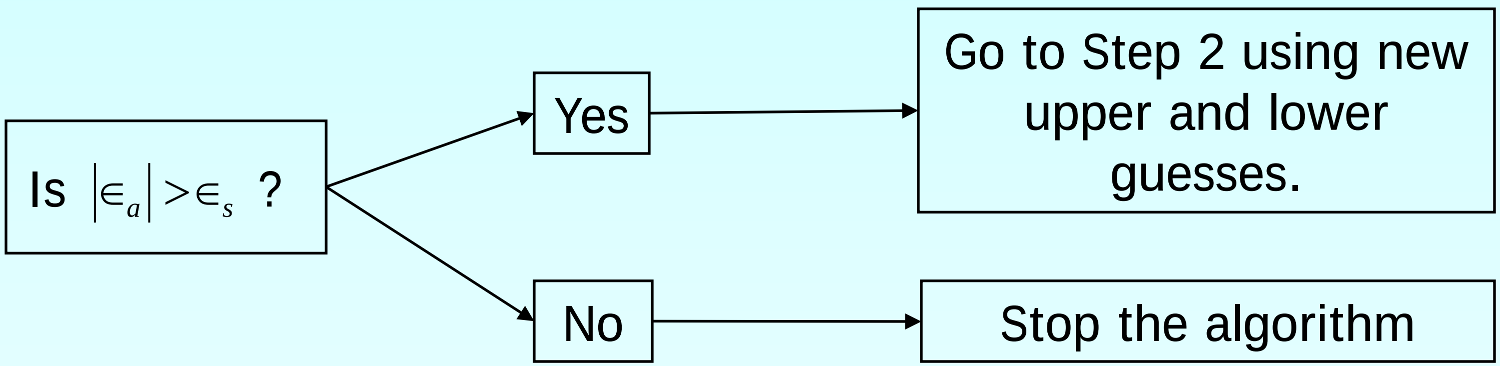
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $\left|\in_a\right|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example 1

Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature.

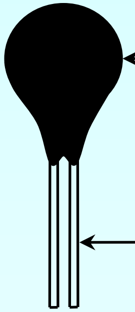


Figure 5 A typical thermistor.

For a 10K3A Betatherm thermistor, the relationship between the resistance, *R*, of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

where T is in Kelvin and R is in ohms.

Example 1 Cont.

For the thermistor, error of no more than $\pm 0.01^{o}$C is acceptable. To find the range of the resistance that is within this acceptable limit at 19^{o}C, we need to solve

$$\frac{1}{19.01+273.15}=1.129241\times10^{-3}+2.341077\times10^{-4}\ln(R)+8.775468\times10^{-8}\{\ln(R)\}^{3}$$

and

$$\frac{1}{18.99+273.15}=1.129241\times10^{-3}+2.341077\times10^{-4}\ln(R)+8.775468\times10^{-8}\{\ln(R)\}^{3}$$

Use the bisection method of finding roots of equations to find the resistance R at 18.99^{o}C.

a) Conduct three iterations to estimate the root of the above equation.

b) Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

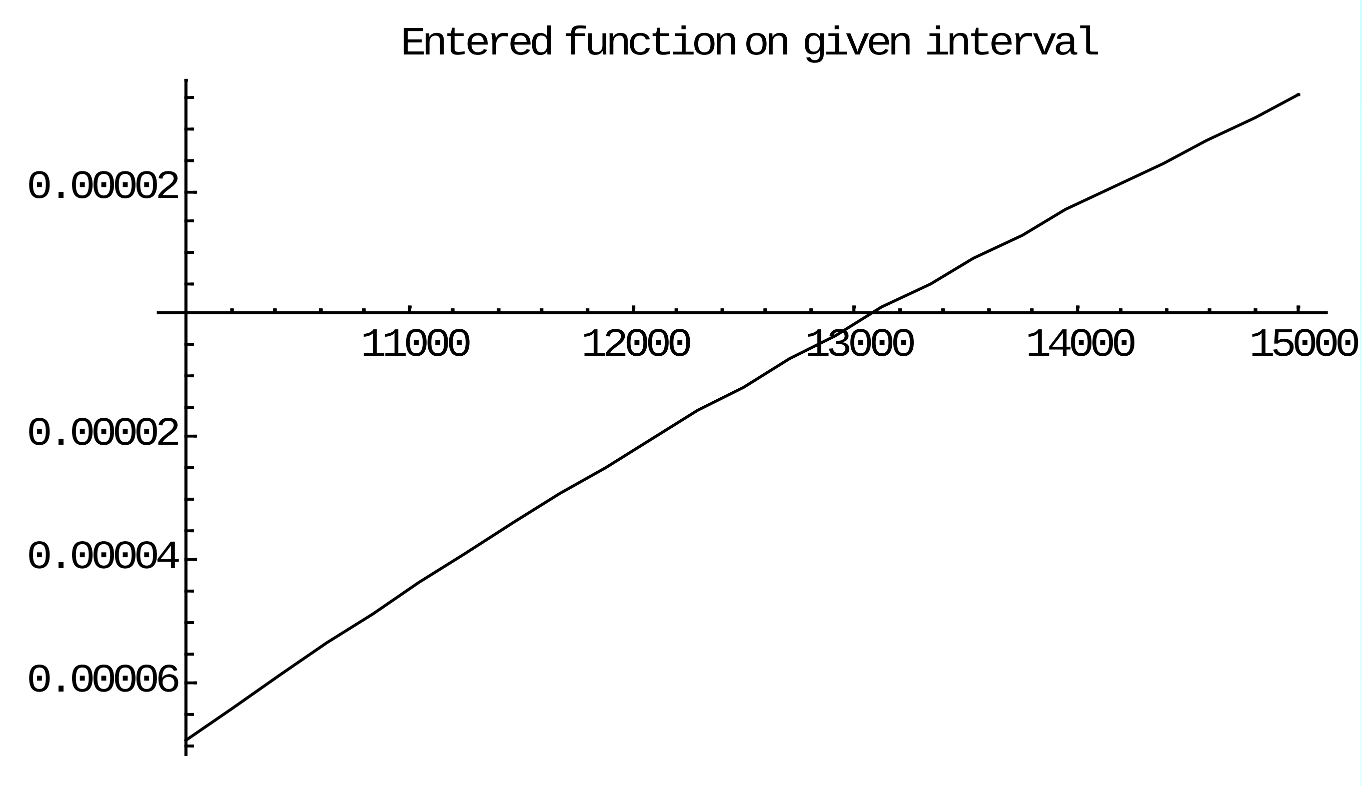


Figure 6 Graph of the function f(R).

$$f(R) = 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3 - 2.293775\times10^{-3}$$

Example 1 Cont.

Solution

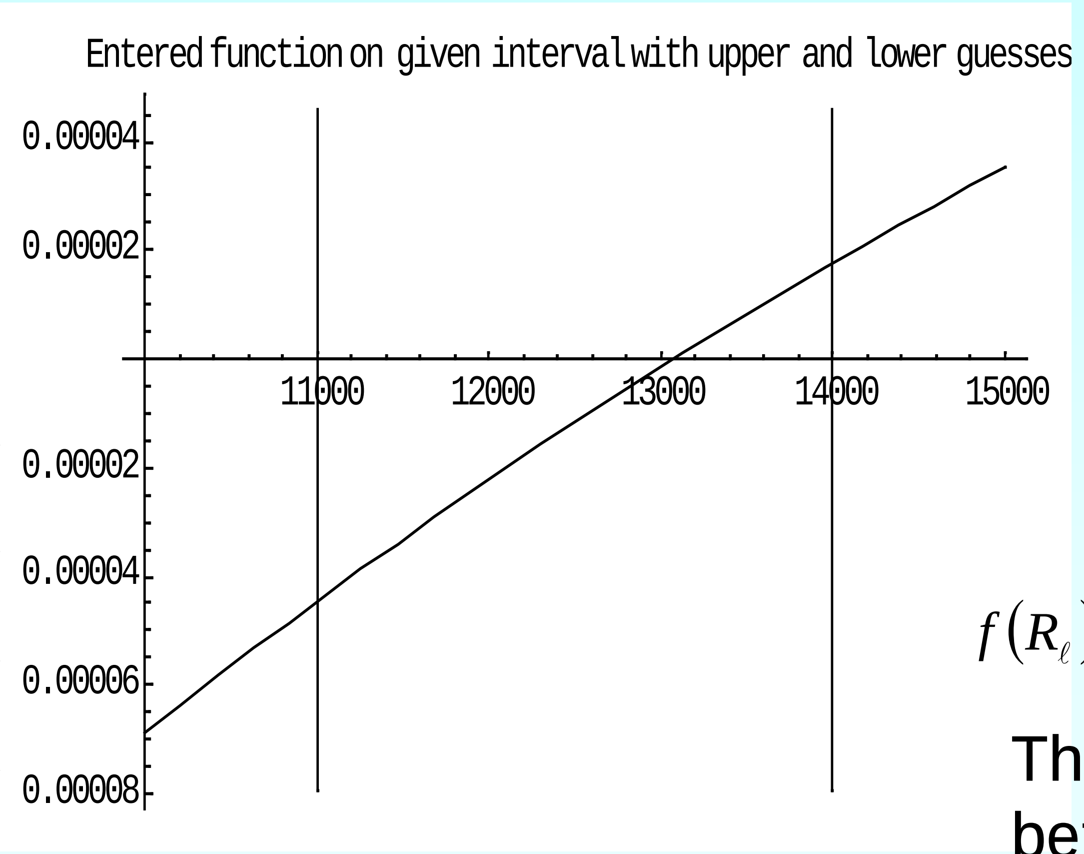


Figure 7 Checking the sign change between the bracket.

Choose the bracket

$$R_\ell = 11000$$

$$R_u = 14000$$

$$f(11000) = -4.4536 \times 10^{-5}$$

$$f(14000) = 1.7563 \times 10^{-5}$$

$$f(R_\ell) f(R_u) = f(11000) f(14000) < 0$$

There is at least one root between R_ℓ and R_u .

Example 1 Cont.

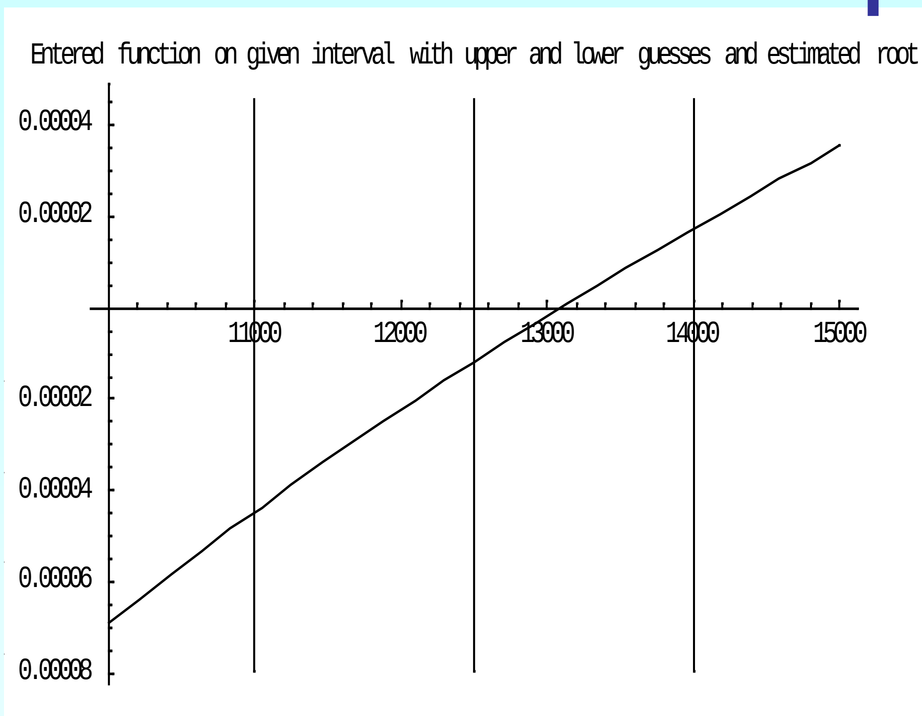


Figure 8 Graph of the estimate of the root after Iteration 1.

<u>Iteration 1</u>

The estimate of the root is

$$R_m = \frac{11000 + 14000}{2} = 12500$$

$$f(12500) = -1.1655 \times 10^{-5}$$

$$f(R_l)f(R_m) = f(11000)f(12500) > 0$$

The root is bracketed between R_m and R_u. The lower and upper limits of the new bracket are

$$R_l = 12500, \; R_u = 14000$$

The absolute relative approximate error cannot be calculated as we do not have a previous approximation.

Example 1 Cont.

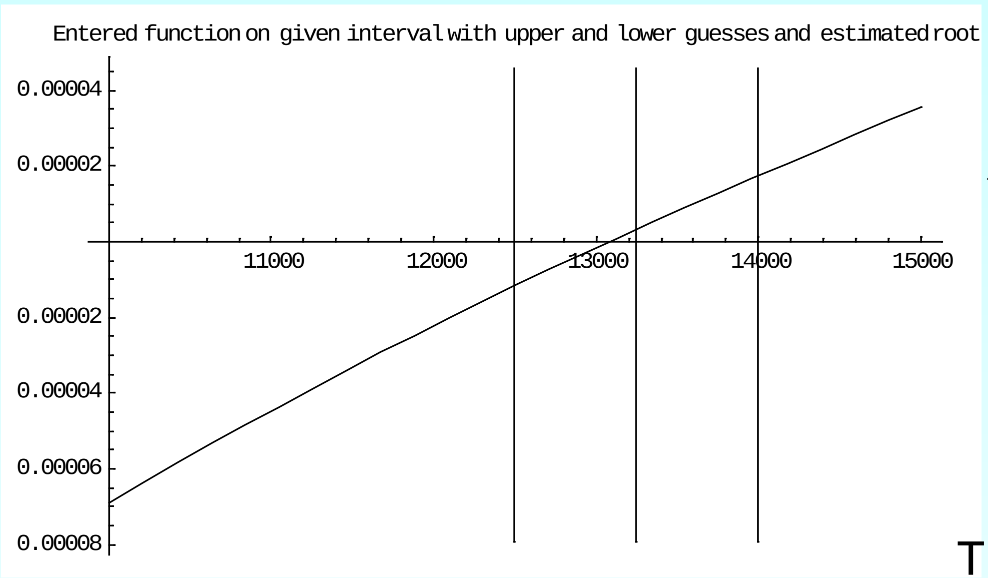


Figure 9 Graph of the estimate of the root after Iteration 2.

<u>Iteration 2</u>

The estimate of the root is

$$R_m = \frac{12500 + 14000}{2} = 13250$$

$$f(13250) = 3.3599 \times 10^{-6}$$

$$f(R_l) f(R_m) = f(12500) f(13250) < 0$$

The root is bracketed between R_l and R_m. The lower and upper limits of the new bracket are

$$R_l = 12500, \; R_u = 13250$$

Example 1 Cont.

The absolute relative approximate error after Iteration 2 is

$$\left|\in_a\right| = \left|\frac{R_m^{new} - R_m^{old}}{R_m^{new}}\right| \times 100$$

$$= \left|\frac{13250 - 12500}{13250}\right| \times 100$$

$$= 5.6604\%$$

None of the significant digits are at least correct in the estimated root as the absolute relative approximate error is greater than 5%.

Example 1 Cont.

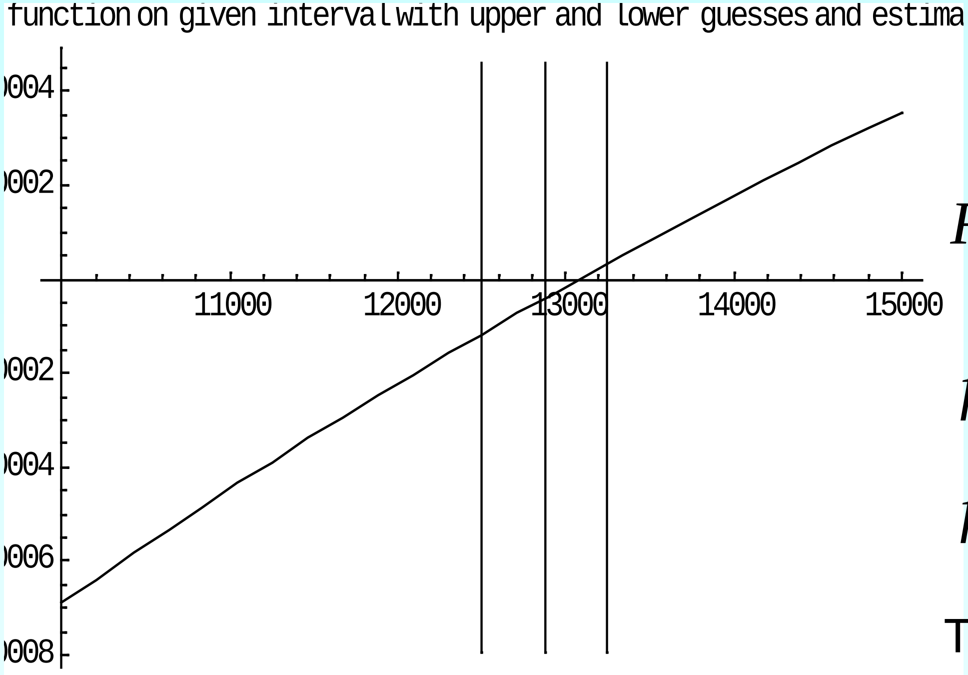


Figure 10 Graph of the estimate of the root after Iteration 3.

Iteration 3

The estimate of the root is

$$R_m = \frac{12500 + 13250}{2} = 12875$$

$$f(12875) = -4.0403 \times 10^{-6}$$

$$f(R_l) f(R)_m = f(12500) f(12875) > 0$$

The root is bracketed between R_m and R_u. The lower and upper limits of the new bracket are

$$R_l = 12875, \; R_u = 13250$$

Example 1 Cont.

The absolute relative approximate error after Iteration 3 is

$$\left|\in_a\right| = \left|\frac{R_m^{new} - R_m^{old}}{R_m^{new}}\right| \times 100$$

$$= \left|\frac{12875 - 13250}{12875}\right| \times 100$$

$$= 2.9126\%$$

The number of significant digits that are at least correct in the estimated root is 1 as the absolute relative approximate error is less than 5%.

Convergence

Table 1 Root of $f(R)$ =0 as function of the number of iterations for bisection method.

Iteration	R_l	R_u	R_m	$\lvert\in_a\rvert$ %	$f(R_m)$
1	11000	14000	12500	----------	1.1655×10^{-5}
2	12500	14000	13250	5.6604	3.3599×10^{-6}
3	12500	13250	12875	2.9126	-4.0403×10^{-6}
4	12875	13250	13063	1.4354	-3.1417×10^{-7}
5	13063	13250	13156	0.71259	1.5293×10^{-6}
6	13063	13156	13109	0.35757	6.0917×10^{-7}
7	13063	13109	13086	0.17910	1.4791×10^{-7}
8	13063	13086	13074	0.089633	-8.3022×10^{-8}
9	13074	13086	13080	0.044796	3.2470×10^{-8}
10	13074	13080	13077	0.022403	-2.5270×10^{-8}

Advantages

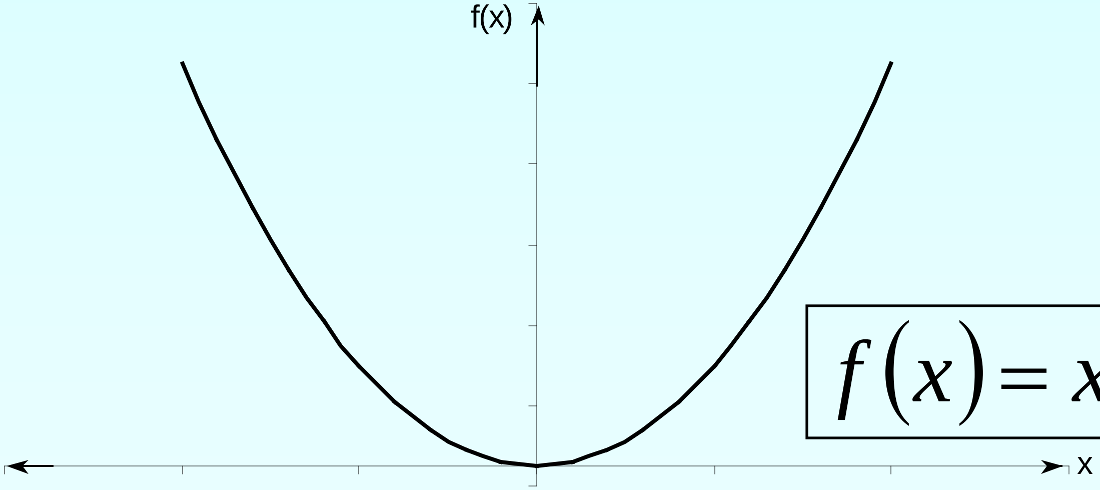
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

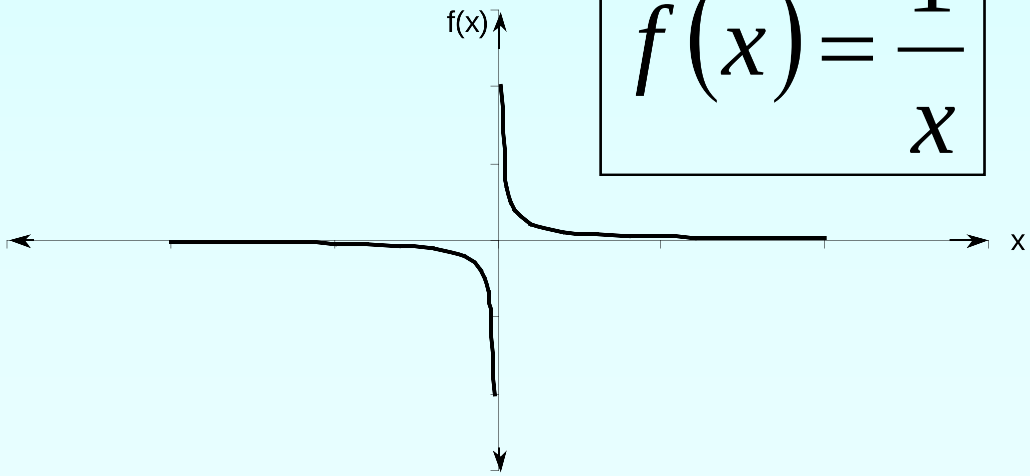
- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist



$$\boxed{f(x)=\frac{1}{x}}$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/bisection_method.html

THE END

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