Chapter 05.03
Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:
1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation, and*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to
(A) evaluate,
(B) differentiate, and
(C) integrate,
relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We will discuss Newton's divided difference polynomial method in this chapter.

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.
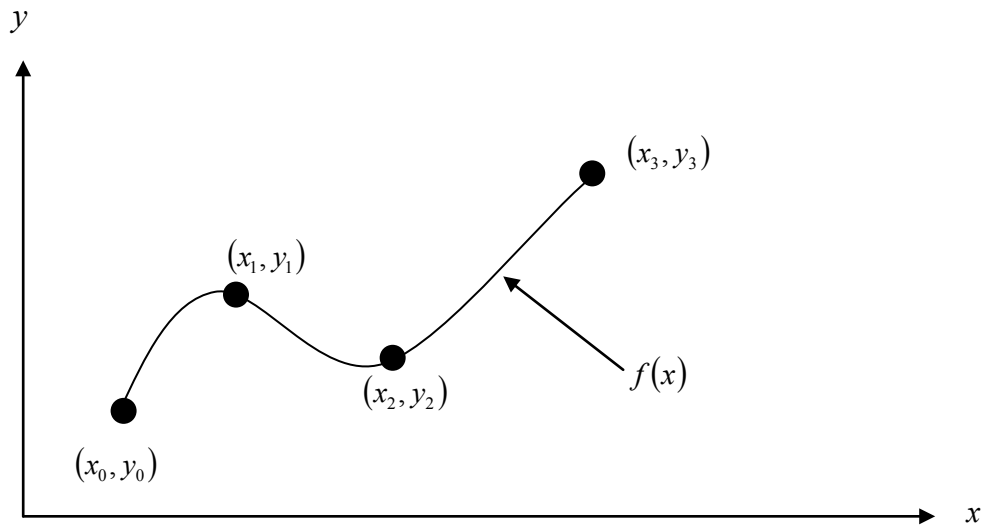
Figure 1   Interpolation of discrete data.

Linear Interpolation
Given (x_0, y_0) and (x_1, y_1), fit a linear interpolant through the data. Noting $y = f(x)$ and $y_1 = f(x_1)$, assume the linear interpolant $f_1(x)$ is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$
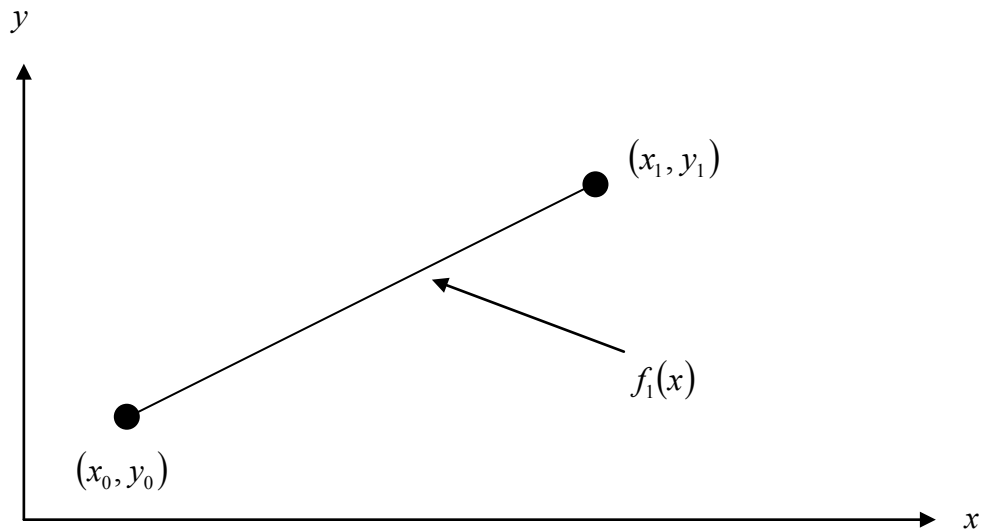$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Figure 2 Linear interpolation.

Example 1

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

Table 1 Temperature as a function of resistance.

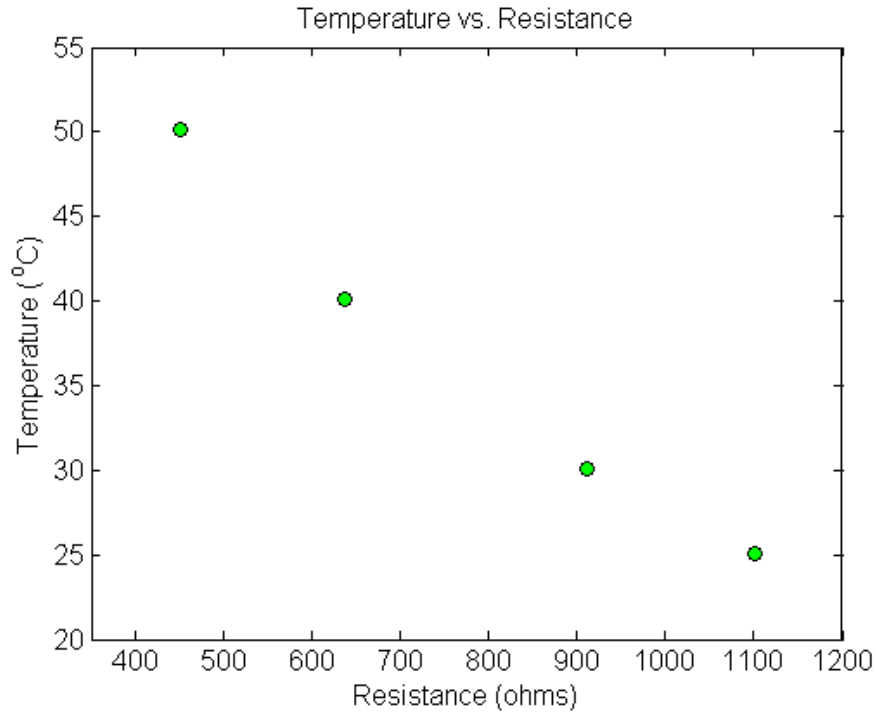| R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

Figure 3 Resistance vs. temperature.

Determine the temperature corresponding to 754.8 ohms using Newton's divided difference method of interpolation and a first order polynomial.

Solution

For linear interpolation, the temperature is given by

$$T(R) = b_0 + b_1(R - R_0)$$

Since we want to find the temperature at $R = 754.8$ and we are using a first order polynomial, we need to choose the two data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The two points are $R = 911.3$ and $R = 636.0$.

Then

$$R_0 = 911.3, \ T(R_0) = 30.131$$
$$R_1 = 636.0, \ T(R_1) = 40.120$$

gives

$$b_0 = T(R_0)$$
$$= 30.131$$
$$b_1 = \frac{T(R_1) - T(R_0)}{R_1 - R_0}$$
$$= \frac{40.120 - 30.131}{636.0 - 911.3}$$
$$= -0.036284$$

Hence

$$T(R) = b_0 + b_1(R - R_0)$$
$$= 30.131 - 0.036284(R - 911.3), \qquad 636.0 \le R \le 911.3$$

At $R = 754.8$
$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$
$$= 35.809\,°C$$

If we expand
$$T(R) = 30.131 - 0.036284(R - 911.3), \qquad 636.0 \le R \le 911.3$$

we get
$$T(R) = 63.197 - 0.036284R, \qquad 636.0 \le R \le 911.3$$

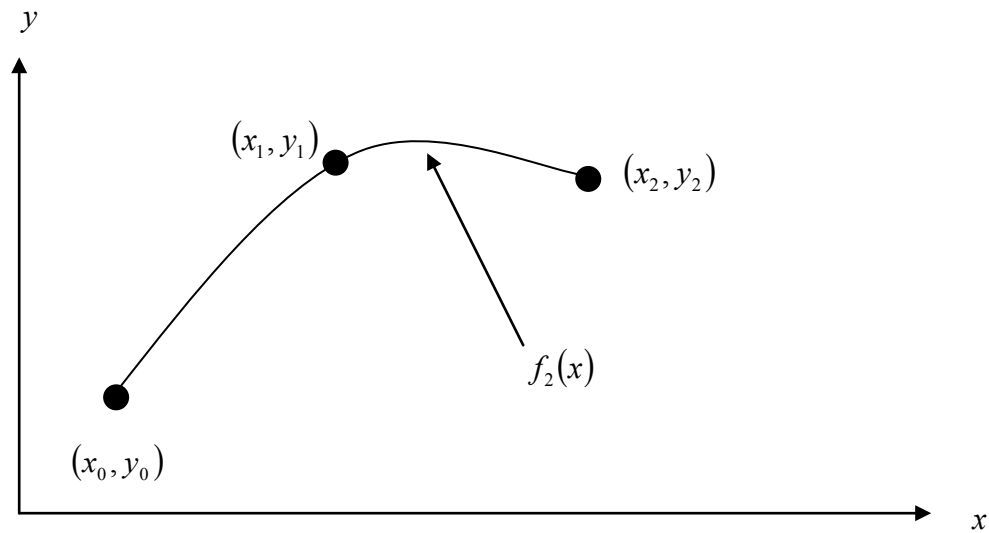This is the same expression that was obtained with the direct method.

Figure 4   Quadratic interpolation.

Example 2

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 2.

Table 2 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

Determine the temperature corresponding to 754.8 ohms using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For quadratic interpolation, the temperature is given by
$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1)$$
Since we want to find the temperature at $R = 754.8$ and we are using a second order polynomial, we need to choose the three data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The three points are $R_0 = 911.3$, $R_1 = 636.0$ and $R_2 = 451.1$.

Then
$$R_0 = 911.3, \ T(R_0) = 30.131$$
$$R_1 = 636.0, \ T(R_1) = 40.120$$
$$R_2 = 451.1, \ T(R_2) = 50.128$$

gives
$$b_0 = T(R_0)$$
$$= 30.131$$
$$b_1 = \frac{T(R_1) - T(R_0)}{R_1 - R_0}$$
$$= \frac{40.120 - 30.131}{636.0 - 911.3}$$
$$= -0.036284$$
$$b_2 = \frac{\dfrac{T(R_2) - T(R_1)}{R_2 - R_1} - \dfrac{T(R_1) - T(R_0)}{R_1 - R_0}}{R_2 - R_0}$$
$$= \frac{\dfrac{50.128 - 40.120}{451.1 - 636.0} - \dfrac{40.120 - 30.131}{636.0 - 911.3}}{451.1 - 911.3}$$
$$= \frac{-0.054127 + 0.036284}{-460.2}$$
$$= 3.8771 \times 10^{-5}$$

Hence
$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1)$$
$$= 30.131 - 0.036284(R - 911.3) + 3.8771 \times 10^{-5}(R - 911.3)(R - 636.0),$$
$$451.1 \le R \le 911.3$$

At $R = 754.8$,
$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$
$$+ 3.8771 \times 10^{-5}(754.8 - 911.3)(754.8 - 636.0)$$
$$= 35.089 \, °C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{35.089 - 35.809}{35.089} \right| \times 100$$

$$= 2.0543\%$$

If we expand,

$$T(R) = 30.131 - 0.036284(R - 911.3) + 3.8771 \times 10^{-5}(R - 911.3)(R - 636.0),$$

$$451.1 \le R \le 911.3$$

we get

$$T(R) = 85.668 - 0.096275R + 3.8771 \times 10^{-5}R^2, \qquad\qquad 451.1 \le R \le 911.3$$

This is the same expression that was obtained with the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that b_0, b_1, and b_2 are finite divided differences. b_0, b_1, and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points, $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots, x_0]$$
$$b_n = f[x_n, x_{n-1}, \ldots, x_0]$$

where the definition of the mth divided difference is

$$b_m = f[x_m, \ldots, x_0]$$
$$= \frac{f[x_m, \ldots, x_1] - f[x_{m-1}, \ldots, x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
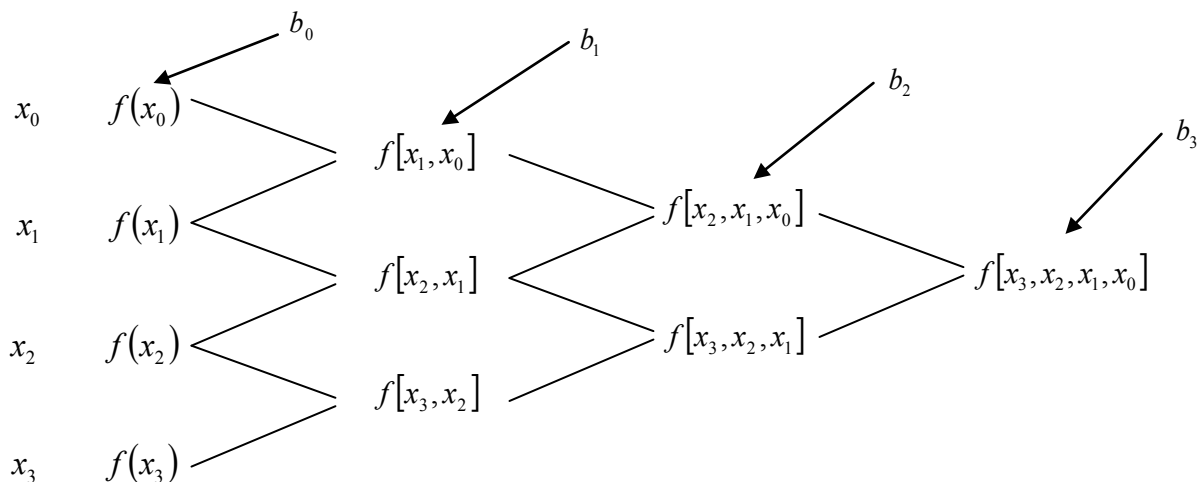$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

Figure 5 Table of divided differences for a cubic polynomial.

Example 3

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance,

you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 3.

Table 3 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

a) Determine the temperature corresponding to 754.8 ohms using Newton's divided difference method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The actual calibration curve used by industry is given by

$$\frac{1}{T} = b_0 + b_1(\ln R - \ln R_0) + b_2(\ln R - \ln R_0)(\ln R - \ln R_1) + b_3(\ln R - \ln R_0)(\ln R - \ln R_1)(\ln R - \ln R_2)$$

substituting $y = \dfrac{1}{T}$, and $x = \ln R$,

the calibration curve is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Table 4 Manipulation for the given data.

R (ohm)	T (°C)	$x\ (\ln R)$	$y\left(\dfrac{1}{T}\right)$
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949

Find the calibration curve and find the temperature corresponding to 754.8 ohms. What is the difference between the results from part (a)? Is the difference larger using results from part (a) or part (b), if the actual measured value at 754.8 ohms is 35.285°C?

Solution

a) For cubic interpolation, the temperature is given by
$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2)$$
Since we want to find the temperature at $R = 754.8$, we need to choose the four data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The four data points are $R_0 = 1101.0$, $R_1 = 911.3$, $R_2 = 636.0$ and $R_3 = 451.1$.
Then
$$R_0 = 1101.0, \quad T(R_0) = 25.113$$

$$R_1 = 911.3, \quad T(R_1) = 30.131$$
$$R_2 = 636.0, \quad T(R_2) = 40.120$$
$$R_3 = 451.1, \quad T(R_3) = 50.128$$

gives

$$b_0 = T[R_0]$$
$$= T(R_0)$$
$$= 25.113$$
$$b_1 = T[R_1, R_0]$$
$$= \frac{T(R_1) - T(R_0)}{R_1 - R_0}$$
$$= \frac{30.131 - 25.113}{911.3 - 1101.0}$$
$$= -0.026452$$
$$b_2 = T[R_2, R_1, R_0]$$
$$= \frac{T[R_2, R_1] - T[R_1, R_0]}{R_2 - R_0}$$
$$T[R_2, R_1] = \frac{T(R_2) - T(R_1)}{R_2 - R_1}$$
$$= \frac{40.120 - 30.131}{636.0 - 911.3}$$
$$= -0.036284$$
$$T[R_1, R_0] = -0.026452$$
$$b_2 = \frac{T[R_2, R_1] - T[R_1, R_0]}{R_2 - R_0}$$
$$= \frac{-0.036284 + 0.026452}{636.0 - 1101.0}$$
$$= 2.1144 \times 10^{-5}$$

$$b_3 = T[R_3, R_2, R_1, R_0]$$
$$= \frac{T[R_3, R_2, R_1] - T[R_2, R_1, R_0]}{R_3 - R_0}$$
$$T[R_3, R_2, R_1] = \frac{T[R_3, R_2] - T[R_2, R_1]}{R_3 - R_1}$$
$$T[R_3, R_2] = \frac{T(R_3) - T(R_2)}{R_3 - R_2}$$
$$= \frac{50.128 - 40.120}{451.1 - 636.0}$$
$$= -0.054127$$
$$T[R_2, R_1] = -0.036284$$

$$T[R_3, R_2, R_1] = \frac{T[R_3, R_2] - T[R_2, R_1]}{R_3 - R_1}$$

$$= \frac{-0.054127 + 0.036284}{451.1 - 911.3}$$

$$= 3.8771 \times 10^{-5}$$

$$T[R_2, R_1, R_0] = 2.1144 \times 10^{-5}$$

$$b_3 = \frac{T[R_3, R_2, R_1] - T[R_2, R_1, R_0]}{R_3 - R_0}$$

$$= \frac{3.8771 \times 10^{-5} - 2.1144 \times 10^{-5}}{451.1 - 1101.0}$$

$$= -2.7124 \times 10^{-8}$$

Hence

$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2)$$

$$= 25.113 - 0.026452(R - 1101.0) + 2.1144 \times 10^{-5}(R - 1101.0)(R - 911.3)$$

$$- 2.7124 \times 10^{-8}(R - 1101.0)(R - 911.3)(R - 636.0), \quad 451.1 \le R \le 1101.0$$

At $R = 754.8$,

$$T(754.8) = 25.113 - 0.026452(754.8 - 1101.0) + 2.1144 \times 10^{-5}(754.8 - 1101.0)(754.8 - 911.3)$$

$$- 2.7124 \times 10^{-8}(754.8 - 1101.0)(754.8 - 911.3)(754.8 - 626.0)$$

$$= 35.242 \,^\circ C$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the second and third order polynomial is

$$|\epsilon_a| = \left| \frac{35.242 - 35.089}{35.242} \right| \times 100$$

$$= 0.43458\%$$

If we expand

$$T(R) = 25.113 - 0.026452(R - 1101.0) + 2.1144 \times 10^{-5}(R - 1101.0)(R - 911.3)$$

$$- 2.7124 \times 10^{-8}(R - 1101.0)(R - 911.3)(R - 636.0), \quad\quad 451.1 \le R \le 1101.0$$

we get

$$T(R) = 92.759 - 0.13093R + 9.2975 \times 10^{-5} R^2 - 2.7124 \times 10^{-8} R^3, \; 451.1 \le R \le 1101.0$$

This is the same expression that was obtained with the direct method.

b) Finding the cubic interpolant using Newton's divided difference for

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

requires that we first calculate the new values of x and y.

x $(\ln R)$	$y\left(\dfrac{1}{T}\right)$
7.0040	0.039820
6.8149	0.033188
6.4552	0.024925
6.1117	0.019949

Then

$$x_0 = 7.0040, \quad y(x_0) = 0.039820$$
$$x_1 = 6.8149, \quad y(x_1) = 0.033188$$
$$x_2 = 6.4552, \quad y(x_2) = 0.024925$$
$$x_3 = 6.1117, \quad y(x_3) = 0.019949$$

gives

$$b_0 = y[x_0]$$
$$= y(x_0)$$
$$= 0.039820$$
$$b_1 = y[x_1, x_0]$$
$$= \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$
$$= \frac{0.033188 - 0.039820}{6.8149 - 7.0040}$$
$$= 0.035069$$
$$b_2 = y[x_2, x_1, x_0]$$
$$= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$
$$y[x_2, x_1] = \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$
$$= \frac{0.024925 - 0.033188}{6.4552 - 6.8149}$$
$$= 0.022974$$
$$y[x_1, x_0] = 0.035069$$

$$b_2 = \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$
$$= \frac{0.022974 - 0.035069}{6.4552 - 7.0040}$$
$$= 0.022040$$

$$b_3 = y[x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_3,x_2,x_1] - y[x_2,x_1,x_0]}{x_3 - x_0}$$

$$y[x_3,x_2,x_1] = \frac{y[x_3,x_2] - y[x_2,x_1]}{x_3 - x_1}$$

$$y[x_3,x_2] = \frac{y(x_3) - y(x_2)}{x_3 - x_2}$$

$$= \frac{0.019949 - 0.024925}{6.1117 - 6.4552}$$

$$= 0.014487$$

$$y[x_2,x_1] = 0.022974$$

$$y[x_3,x_2,x_1] = \frac{y[x_3,x_2] - y[x_2,x_1]}{x_3 - x_1}$$

$$= \frac{0.014487 - 0.022974}{6.1117 - 6.8149}$$

$$= 0.012070$$

$$y[x_2,x_1,x_0] = 0.022040$$

$$b_3 = \frac{y[x_3,x_2,x_1] - y[x_2,x_1,x_0]}{x_3 - x_0}$$

$$= \frac{0.012070 - 0.022040}{6.1117 - 7.0040}$$

$$= 0.011173$$

Hence

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$= 0.039820 + 0.035069(x - 7.0040) + 0.022974(x - 7.0040)(x - 6.8149)$$
$$+ 0.011173(x - 7.0040)(x - 6.8149)(x - 6.4552), \quad 6.1117 \le x \le 7.0040$$

Since we're looking for the temperature at $R = 754.8$, we will be using

$$x = \ln(754.8)$$
$$= 6.6265$$

At $x = 6.6265$,

$$y(6.6265) = 0.039820 + 0.035071(6.6265 - 7.0040)$$
$$+ 0.022972(6.6265 - 7.0040)(6.6265 - 6.8149)$$
$$+ 0.011182(6.6265 - 7.0040)(6.6265 - 6.8149)(6.6265 - 6.4552)$$
$$= 0.028285$$

Finally, since $y = \dfrac{1}{T}$,

$$T = \frac{1}{y}$$

$$= \frac{1}{0.028285}$$

$$= 35.355\,^\circ\text{C}$$

Since the actual measured value at 754.8 ohms is $35.285\,^\circ\text{C}$, the absolute relative true error for the value found in part (a) is

$$\left|\epsilon_t\right| = \left|\frac{35.285 - 35.242}{35.285}\right| \times 100$$

$$= 0.12253\%$$

and for part (b) is

$$\left|\epsilon_t\right| = \left|\frac{35.285 - 35.355}{35.285}\right| \times 100$$

$$= 0.19825\%$$

Therefore, the cubic polynomial interpolant given by Newton's divided difference method, that is,

$$T(R) = b_0 + b_1(R - R_0) + b_2(R - R_0)(R - R_1) + b_3(R - R_0)(R - R_1)(R - R_2)$$

obtained more accurate results than the calibration curve of

$$\frac{1}{T} = b_0 + b_1(\ln R - \ln R_0) + b_2(\ln R - \ln R_0)(\ln R - \ln R_1) + b_3(\ln R - \ln R_0)(\ln R - \ln R_1)(\ln R - \ln R_2)$$

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Examples of Newton's divided difference interpolation.
Major	Electrical Engineering
Authors	Autar Kaw
Date	November 14, 2012
Web Site	http://numericalmethods.eng.usf.edu