Runge 4th Order Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$$

$$k_4 = f(x_i + h, y_i + k_3h)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A rectifier-based power supply requires a capacitor to temporarily store power when the rectified waveform from the AC source drops below the target voltage. To properly size this capacitor a first-order ordinary differential equation must be solved. For a particular power supply, with a capacitor of 150 μF, the ordinary differential equation to be solved is

$$\frac{dv(t)}{dt} = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v(t)}{0.04},0\right)\right\} \qquad v(0)=0$$

Find voltage across the capacitor at t= 0.00004s. Use step size h=0.00002

$$\frac{dv}{dt} = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$f(t,v) = \frac{1}{150\times10^{-6}}\left\{-0.1+\max\left(\frac{\left|18\cos(120\pi(t))\right|-2-v}{0.04},0\right)\right\}$$

$$v_{i+1} = v_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

Solution

Step 1: $i = 0,\ t_0 = 0,\ v_0 = 0V$

$$k_1 = f\left(t_0, v_0\right) = f\left(0,\ 0\right) = \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos\left(120\pi(0)\right)\right| - 2 - (0)}{0.04}, 0\right)\right\} = 2.6660\times10^{6}$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(0.00002),\ 0 + \frac{1}{2}\left(2.6660\times10^{6}\right)0.00002\right) = f\left(0.00001,\ 26.660\right)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos\left(120\pi(0.00001)\right)\right| - 2 - (26.660)}{0.04}, 0\right)\right\} = -666.67$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, v_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(0.00002),\ 0 + \frac{1}{2}(-666.67)0.00002\right) = f\left(0.00001,\ -0.0066667\right)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos\left(120\pi(0.00001)\right)\right| - 2 - (-0.0066667)}{0.04}, 0\right)\right\} = 2.6671\times10^{6}$$

$$k_4 = f\left(t_0 + h, v_0 + k_3 h\right) = f\left(0 + (0.0002),\ 0 + \left(2.6671\times10^{6}\right)0.00002\right) = f\left(0.00002,\ 53.342\right)$$

$$= \frac{1}{150\times10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos\left(120\pi(0.00002)\right)\right| - 2 - (53.342)}{0.04}, 0\right)\right\} = -666.67$$

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Solution Cont

$$v_1 = v_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 0 + \frac{1}{6}\left(2.6660\times10^6 + 2\left(-666.67\right) + 2\left(2.6671\times10^6\right) + \left(-666.67\right)\right)0.00002$$

$$= 0 + \frac{1}{6}\left(7.9982\times10^6\right)0.00002$$

$$= 26.661\text{V}$$

v_1 is the approximate voltage at

$$t = t_1 = t_0 + h = 0 + 0.00002 = 0.00002$$

$$v\left(0.00002\right) \approx v_1 = 26.661\text{V}$$

Solution Cont

Step 2: $i = 1, t_1 = 0.00002, v_1 = 26.641V$

$$k_1 = f(t_1, v_1) = f(0.00002,\ 26.661) = \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00002))\right| - 2 - (26.661)}{0.04}, 0\right)\right\} = -666.67$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_1 h\right) = f\left(0.00002 + \frac{1}{2}(0.00002),\ 26.661 + \frac{1}{2}(-666.67)0.00002\right) = f(0.00003,\ 26.654)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.654)}{0.04}, 0\right)\right\} = -666.67$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, v_1 + \frac{1}{2}k_2 h\right) = f\left(0.00002 + \frac{1}{2}(0.00002),\ 26.661 + \frac{1}{2}(-666.67)0.00002\right) = f(0.00003,\ 26.654)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.654)}{0.04}, 0\right)\right\} = -666.67$$

$$k_4 = f(t_1 + h, v_1 + k_3 h) = f(0.00002 + (0.00002),\ 26.661 + (-666.67)0.00002) = f(0.00003,\ 26.647)$$

$$= \frac{1}{150 \times 10^{-6}}\left\{-0.1 + \max\left(\frac{\left|18\cos(120\pi(0.00003))\right| - 2 - (26.634)}{0.04}, 0\right)\right\} = -666.67$$

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Solution Cont

$$v_2 = v_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 26.661 + \frac{1}{6}\left(-666.67 + 2\left(-666.67\right) + 2\left(-666.67\right) + \left(-666.67\right)\right)0.00002$$

$$= 26.661 + \frac{1}{6}\left(-4000.0\right)0.00002$$

$$= 26.647\text{V}$$

v_2 is the approximate voltage at

$$t_2 = t_1 + h = 0.00002 + 0.00002 = 0.00004\,s$$

$$v\left(0.00004\right) \approx v_2 = 26.647V$$

Solution Cont

The **exact** solution to the differential equation at t=0.00004 seconds is

$$v(0.00004) = 15.974V$$

Comparison with exact results

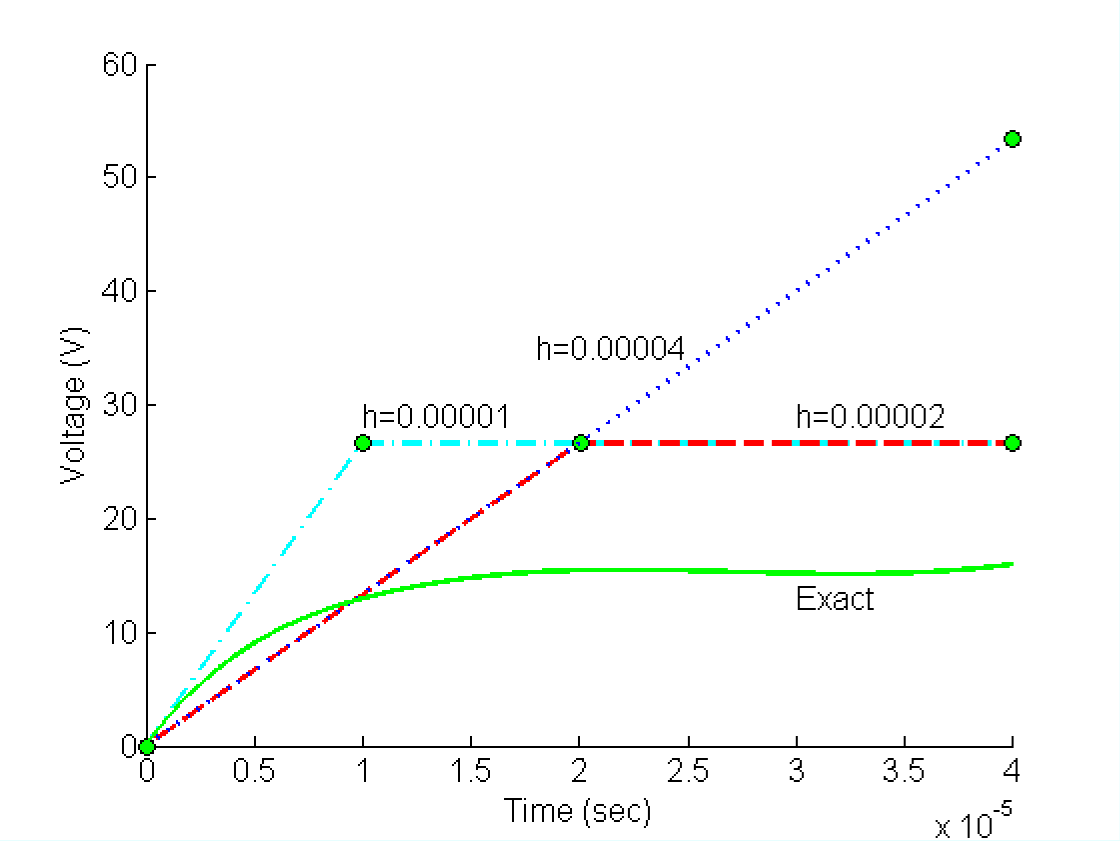

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1 Value of voltage at time, t=0.00004s for different step sizes

Step size, h	$v(0.00004)$	E_t	$\lvert\in_t\rvert$ %
0.00004	53.335	−37.361	233.89
0.00002	26.647	−10.673	66.817
0.00001	15.986	−0.012299	0.076996
0.000005	15.975	−0.00050402	0.0031552
0.0000025	15.976	−0.0015916	0.0099639

$v(0.00004) = 15.974V$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

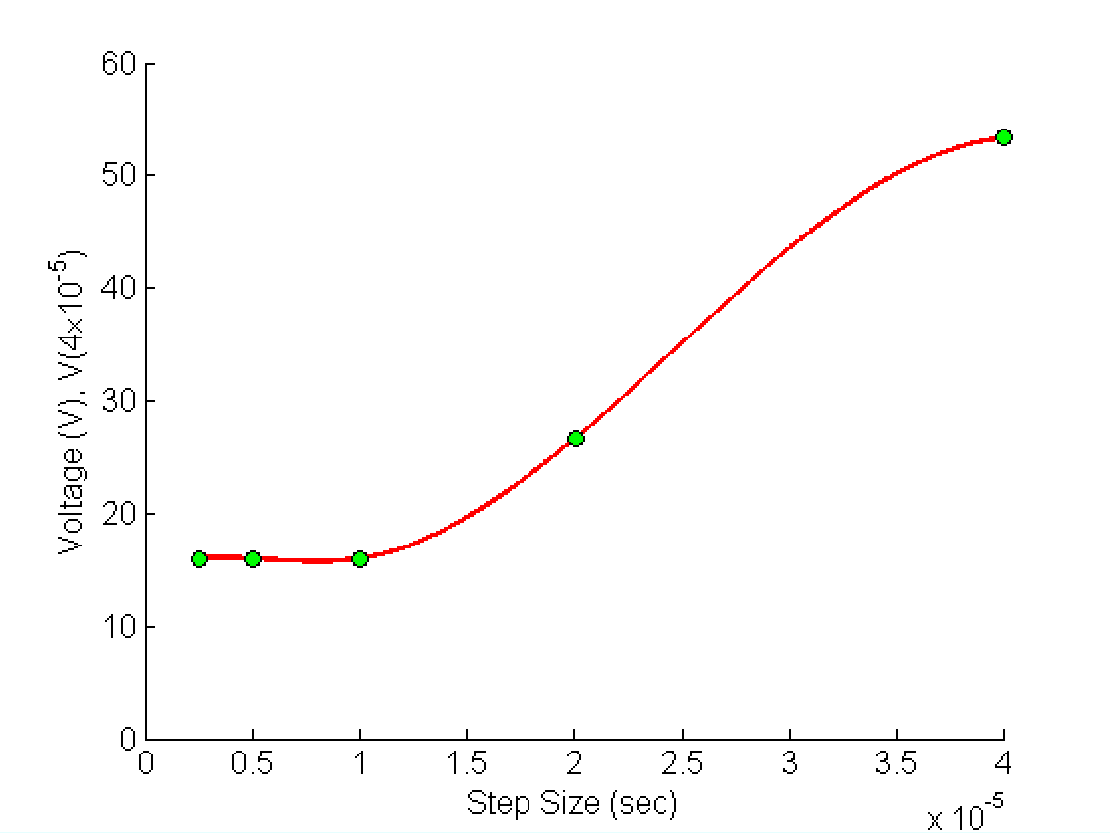

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

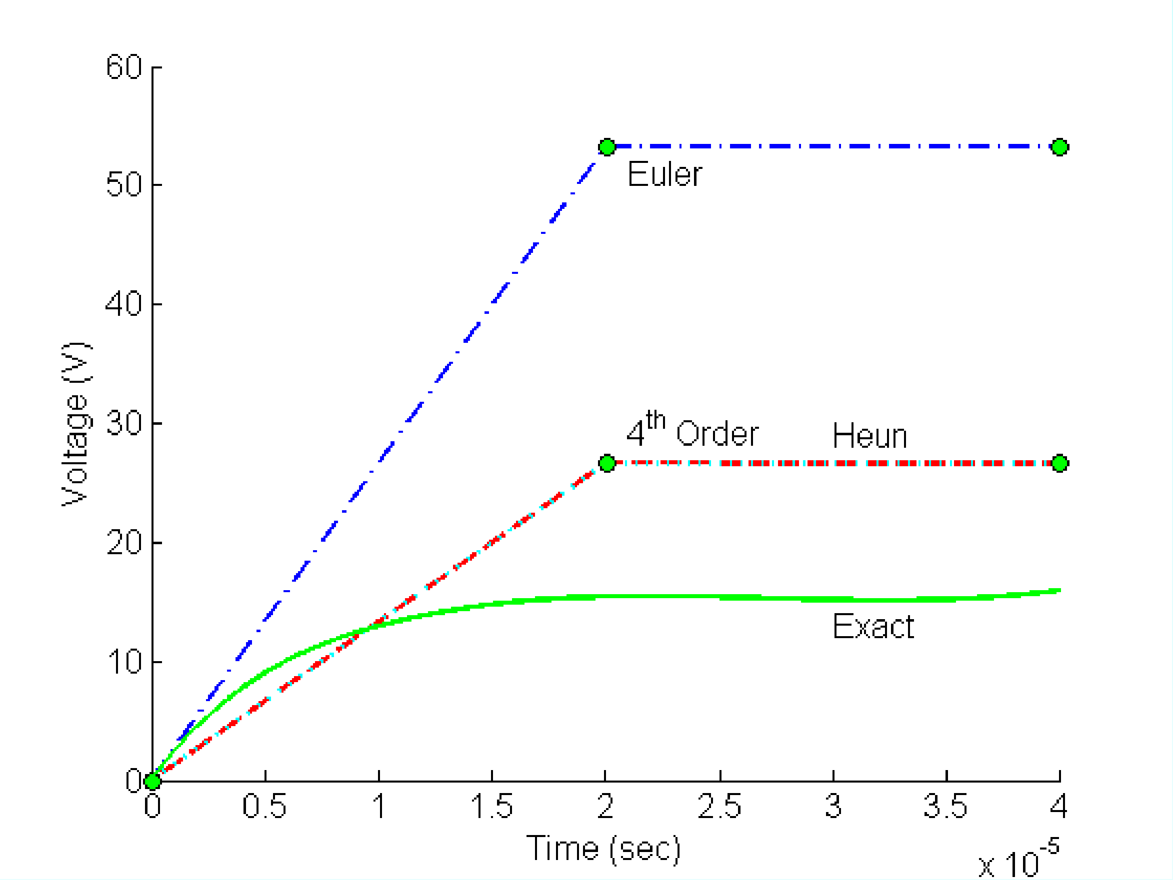

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

THE END

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