Measuring Errors

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates
Measuring Errors

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Why measure errors?

1) To determine the accuracy of numerical results.
2) To develop stopping criteria for iterative algorithms.
True Error

- Defined as the difference between the true value in a calculation and the approximate value found using a numerical method etc.

True Error = True Value − Approximate Value
Example—True Error

The derivative, \( f'(x) \) of a function \( f(x) \) can be approximated by the equation,

\[
f'(x) \approx \frac{f(x + h) - f(x)}{h}
\]

If \( f(x) = 7e^{0.5x} \) and \( h = 0.3 \)

a) Find the approximate value of \( f'(2) \)

b) True value of \( f'(2) \)

c) True error for part (a)
Example (cont.)

Solution:

a) For $x = 2$ and $h = 0.3$

\[
f'(2) \approx \frac{f(2 + 0.3) - f(2)}{0.3}
\]

\[
= \frac{f(2.3) - f(2)}{0.3}
\]

\[
= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}
\]

\[
= \frac{22.107 - 19.028}{0.3} = 10.263
\]
Example (cont.)

Solution:

b) The exact value of $f''(2)$ can be found by using our knowledge of differential calculus.

$$f(x) = 7e^{0.5x}$$

$$f'(x) = 7 \times 0.5 \times e^{0.5x}$$

$$= 3.5e^{0.5x}$$

So the true value of $f''(2)$ is

$$f''(2) = 3.5e^{0.5(2)}$$

$$= 9.5140$$

True error is calculated as

$$E_t = \text{True Value} - \text{Approximate Value}$$

$$= 9.5140 - 10.263 = -0.722$$
Relative True Error

- Defined as the ratio between the true error, and the true value.

Relative True Error ($\varepsilon_r$) = \frac{\text{True Error}}{\text{True Value}}
Example—Relative True Error

Following from the previous example for true error, find the relative true error for \( f(x) = 7e^{0.5x} \) at \( f'(2) \) with \( h = 0.3 \)

From the previous example,
\[
E_t = -0.722
\]

Relative True Error is defined as
\[
\varepsilon_t = \frac{\text{True Error}}{\text{True Value}} = \frac{-0.722}{9.5140} = -0.075888
\]
as a percentage,
\[
\varepsilon_t = -0.075888 \times 100\% = -7.5888\%
\]
Approximate Error

- What can be done if true values are not known or are very difficult to obtain?
- Approximate error is defined as the difference between the present approximation and the previous approximation.

**Approximate Error** ($E_a$) = Present Approximation - Previous Approximation
Example—Approximate Error

For \( f(x) = 7e^{0.5x} \) at \( x = 2 \) find the following,

a) \( f'(2) \) using \( h = 0.3 \)

b) \( f'(2) \) using \( h = 0.15 \)

c) approximate error for the value of \( f'(2) \) for part b)

Solution:

a) For \( x = 2 \) and \( h = 0.3 \)

\[
f'(x) \approx \frac{f(x + h) - f(x)}{h}
\]

\[
f'(2) \approx \frac{f(2 + 0.3) - f(2)}{0.3}
\]
Example (cont.)

Solution: (cont.)

\[ f'(2) \approx \frac{f(2 + 0.15) - f(2)}{0.15} \]

\[ f'(2) \approx \frac{f(2.15) - f(2)}{0.15} \]

b) For \( x = 2 \) and \( h = 0.15 \)

\[ f'(2) \approx \frac{f(2 + 0.15) - f(2)}{0.15} \]

\[ f'(2) \approx \frac{f(2.15) - f(2)}{0.15} \]
Example (cont.)

Solution: (cont.)

\[
7e^{0.5(2.15)} - 7e^{0.5(2)} = \frac{7e^{0.5(2.15)} - 7e^{0.5(2)}}{0.15} = \frac{20.50 - 19.028}{0.15} = 9.8800
\]

c) So the approximate error, \( E_a \) is

\[
E_a = \text{Present Approximation} - \text{Previous Approximation}
\]

\[
E_a = 9.8800 - 10.263 = -0.38300
\]
Relative Approximate Error

- Defined as the ratio between the approximate error and the present approximation.

Relative Approximate Error \( (\varepsilon_a) = \frac{\text{Approximate Error}}{\text{Present Approximation}} \)
Example—Relative Approximate Error

For $f(x) = 7e^{0.5x}$ at $x = 2$, find the relative approximate error using values from $h = 0.3$ and $h = 0.15$

Solution:
From Example 3, the approximate value of $f'(2) = 10.263$
using $h = 0.3$ and $f'(2) = 9.8800$ using $h = 0.15$

$$E_a = \text{Present Approximation} - \text{Previous Approximation}$$

$$= 9.8800 - 10.263$$

$$= -0.38300$$
Example (cont.)

Solution: (cont.)

\[ \varepsilon_a = \frac{\text{Approximate Error}}{\text{Present Approximation}} \]

\[ = \frac{-0.38300}{9.8800} = -0.038765 \]

as a percentage,

\[ \varepsilon_a = -0.038765 \times 100\% = -3.8765\% \]

Absolute relative approximate errors may also need to be calculated,

\[ |\varepsilon_a| = |-0.038765| = 0.038765 \text{ or } 3.8765\% \]
How is Absolute Relative Error used as a stopping criterion?

If $|\varepsilon_a| \leq \varepsilon_s$ where $\varepsilon_s$ is a pre-specified tolerance, then no further iterations are necessary and the process is stopped.

If at least $m$ significant digits are required to be correct in the final answer, then

$$|\varepsilon_a| \leq 0.5 \times 10^{2-m}\%$$
Table of Values

For \( f(x) = 7e^{0.5x} \) at \( x = 2 \) with varying step size, \( h \)

| \( h \)  | \( f'(2) \)  | \( | \varepsilon_a | \)  | \( m \)  |
|---------|-------------|----------------|--------|
| 0.3     | 10.263      | N/A            | 0      |
| 0.15    | 9.8800      | 3.877%         | 1      |
| 0.10    | 9.7558      | 1.273%         | 1      |
| 0.01    | 9.5378      | 2.285%         | 1      |
| 0.001   | 9.5164      | 0.2249%        | 2      |
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/measuring_errors.html
THE END

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