

# Sources of Error

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM  
Undergraduates

# Sources of Error

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# Two sources of numerical error

- 1) Round off error
- 2) Truncation error

# Round-off Error

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# Round off Error

- Caused by representing a number approximately

$$\frac{1}{3} \cong 0.333333$$

$$\sqrt{2} \cong 1.4142\dots$$

# Problems created by round off error

- 28 Americans were killed on February 25, 1991 by an Iraqi Scud missile in Dhahran, Saudi Arabia.
- The patriot defense system failed to track and intercept the Scud. Why?

# Problem with Patriot missile

- Clock cycle of 1/10 seconds was represented in 24-bit fixed point register created an error of  $9.5 \times 10^{-8}$  seconds.
- The battery was on for 100 consecutive hours, thus causing an inaccuracy of

$$\begin{aligned} &= 9.5 \times 10^{-8} \frac{\text{s}}{0.1\text{s}} \times 100\text{hr} \times \frac{3600\text{s}}{1\text{hr}} \\ &= 0.342\text{s} \end{aligned}$$



# Problem (cont.)

- The shift calculated in the ranging system of the missile was 687 meters.
- The target was considered to be out of range at a distance greater than 137 meters.



# Effect of Carrying Significant Digits in Calculations

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Find the contraction in the diameter

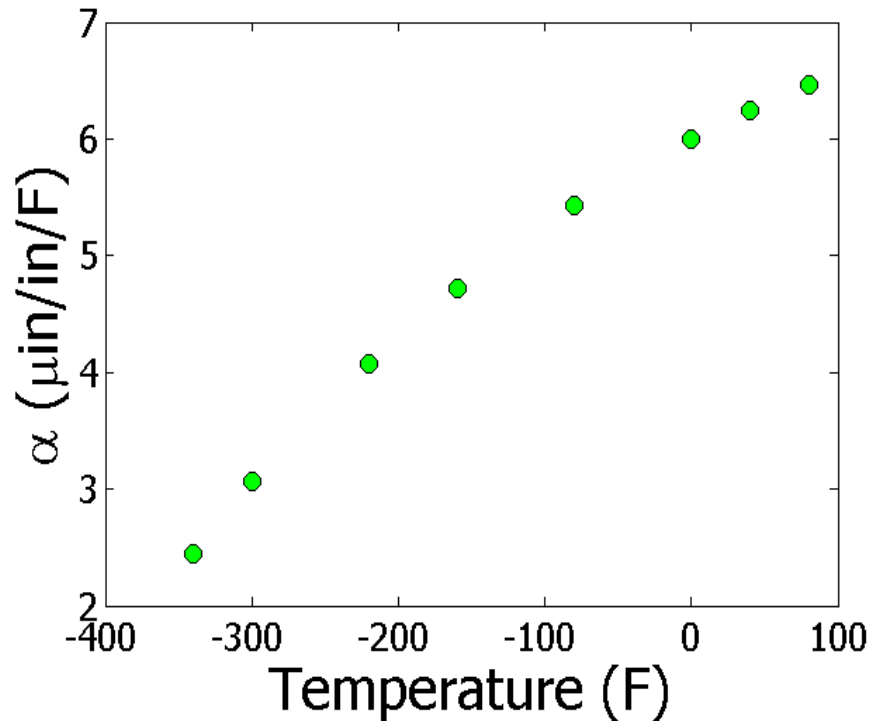


$$\Delta D = D \int_{T_a}^{T_c} \alpha(T) dT$$

$$T_a = 80^\circ\text{F}; T_c = -108^\circ\text{F}; D = 12.363''$$

$$\alpha = a_0 + a_1 T + a_2 T^2$$

# Thermal Expansion Coefficient vs Temperature



T( $^{\circ}\text{F}$ )	$\alpha$ ( $\mu\text{in/in}/^{\circ}\text{F}$ )
-340	2.45
-300	3.07
-220	4.08
-160	4.72
-80	5.43
0	6.00
40	6.24
80	6.47

# Regressing Data in Excel (general format)

$$\alpha = -1\text{E-}05T^2 + 0.0062T + 6.0234$$

# Observed and Predicted Values

$$\alpha = -1\text{E-}05T^2 + 0.0062T + 6.0234$$

T(°F)	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Given	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Predicted
-340	2.45	2.76
-300	3.07	3.26
-220	4.08	4.18
-160	4.72	4.78
-80	5.43	5.46
0	6.00	6.02
40	6.24	6.26
80	6.47	6.46

# Regressing Data in Excel (scientific format)

$$\alpha = -1.2360\text{E-}05T^2 + 6.2714\text{E-}03T + 6.0234$$

# Observed and Predicted Values

$$\alpha = -1.2360\text{E-}05T^2 + 6.2714\text{E-}03T + 6.0234$$

T(°F)	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Given	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Predicted
-340	2.45	2.46
-300	3.07	3.03
-220	4.08	4.05
-160	4.72	4.70
-80	5.43	5.44
0	6.00	6.02
40	6.24	6.25
80	6.47	6.45

# Observed and Predicted Values

$$\alpha = -1.2360\text{E-}05T^2 + 6.2714\text{E-}03T + 6.0234$$

$$\alpha = -1\text{E-}05T^2 + 0.0062T + 6.0234$$

T(°F)	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Given	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Predicted	$\alpha$ ( $\mu\text{in/in/}^\circ\text{F}$ ) Predicted
-340	2.45	2.46	2.76
-300	3.07	3.03	3.26
-220	4.08	4.05	4.18
-160	4.72	4.70	4.78
-80	5.43	5.44	5.46
0	6.00	6.02	6.02
40	6.24	6.25	6.26
80	6.47	6.45	6.46



**THE END**

# Truncation Error

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# Truncation error

- Error caused by truncating or approximating a mathematical procedure.

# Example of Truncation Error

Taking only a few terms of a Maclaurin series to approximate  $e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

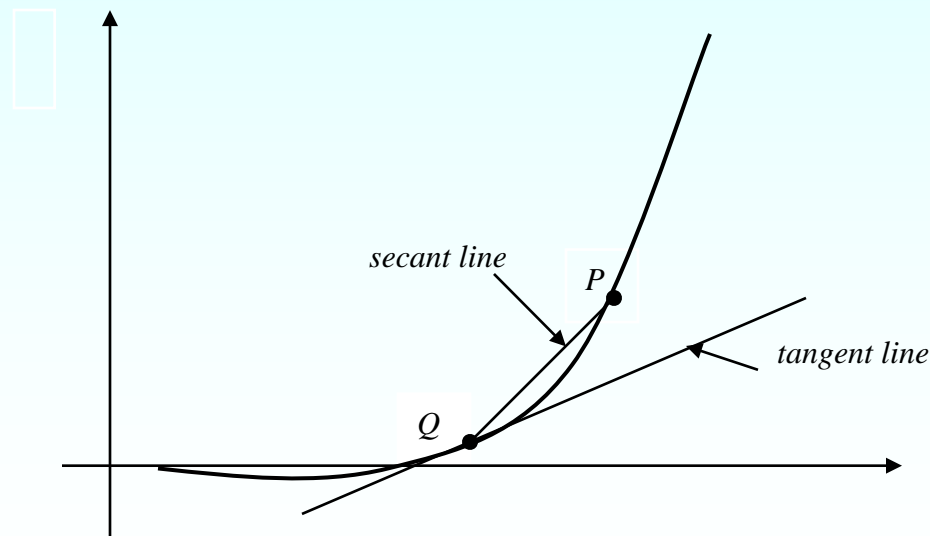
If only 3 terms are used,

$$\text{Truncation Error} = e^x - \left( 1 + x + \frac{x^2}{2!} \right)$$

# Another Example of Truncation Error

Using a finite  $\Delta x$  to approximate  $f'(x)$

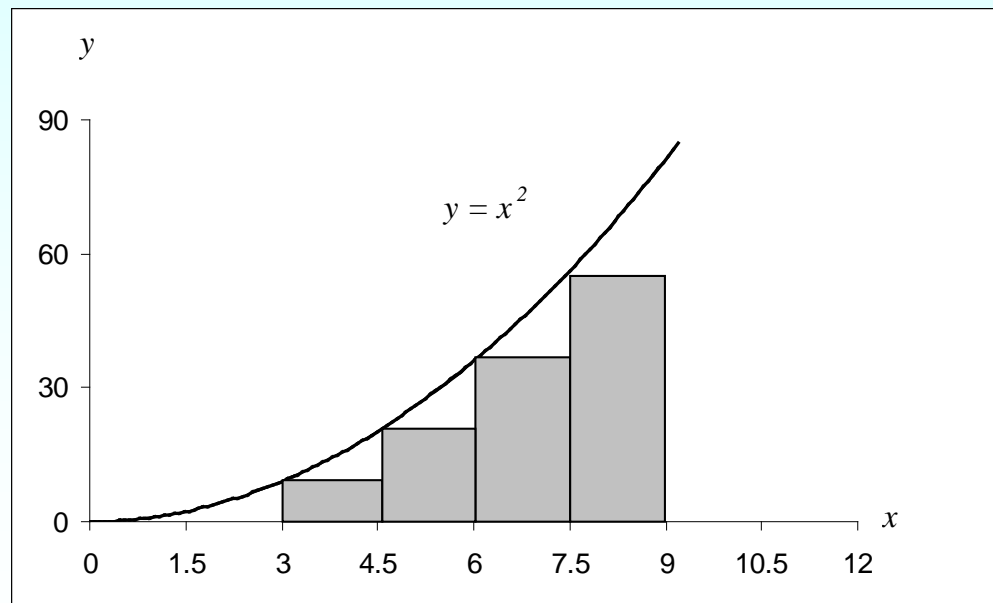
$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



**Figure 1.** Approximate derivative using finite  $\Delta x$

# Another Example of Truncation Error

Using finite rectangles to approximate an integral.



# Example 1 —Maclaurin series

Calculate the value of  $e^{1.2}$  with an absolute relative approximate error of less than 1%.

$$e^{1.2} = 1 + 1.2 + \frac{1.2^2}{2!} + \frac{1.2^3}{3!} + \dots$$

$n$	$e^{1.2}$	$E_a$	$ \epsilon_a \%$
1	1	—	—
2	2.2	1.2	54.545
3	2.92	0.72	24.658
4	3.208	0.288	8.9776
5	3.2944	0.0864	2.6226
6	3.3151	0.020736	0.62550

6 terms are required. How many are required to get at least 1 significant digit correct in your answer?

# Example 2 — Differentiation

Find  $f'(3)$  for  $f(x) = x^2$  using  $f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$   
and  $\Delta x = 0.2$

$$\begin{aligned} f'(3) &= \frac{f(3 + 0.2) - f(3)}{0.2} \\ &= \frac{f(3.2) - f(3)}{0.2} = \frac{3.2^2 - 3^2}{0.2} = \frac{10.24 - 9}{0.2} = \frac{1.24}{0.2} = 6.2 \end{aligned}$$

The actual value is

$$f'(x) = 2x, \quad f'(3) = 2 \times 3 = 6$$

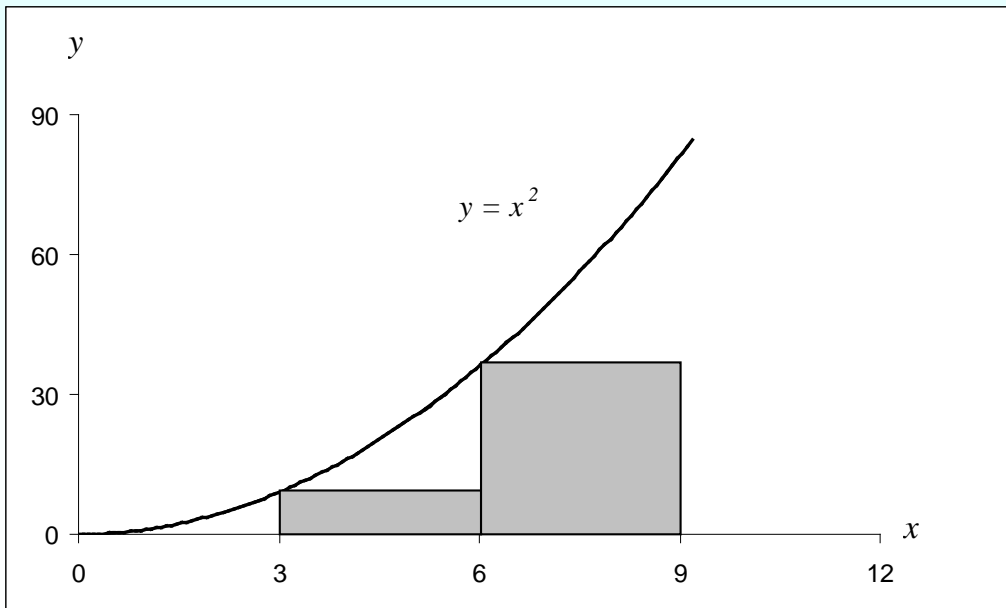
Truncation error is then,  $6 - 6.2 = -0.2$

Can you find the truncation error with  $\Delta x = 0.1$  <sup>24</sup>



# Example 3 — Integration

Use two rectangles of equal width to approximate the area under the curve for  $f(x) = x^2$  over the interval  $[3,9]$



$$\int_3^9 x^2 dx$$

# Integration example (cont.)

Choosing a width of 3, we have

$$\begin{aligned}\int_3^9 x^2 dx &= (x^2)\Big|_{x=3}^{6-3} + (x^2)\Big|_{x=6}^{9-6} \\ &= (3^2)3 + (6^2)3 \\ &= 27 + 108 = 135\end{aligned}$$

Actual value is given by

$$\int_3^9 x^2 dx = \left[ \frac{x^3}{3} \right]_3^9 = \left[ \frac{9^3 - 3^3}{3} \right] = 234$$

Truncation error is then

$$234 - 135 = 99$$

Can you find the truncation error with 4 rectangles?

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

[http://numericalmethods.eng.usf.edu/topics/sources\\_of\\_error.html](http://numericalmethods.eng.usf.edu/topics/sources_of_error.html)

**THE END**

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