Differentiation-Discrete Functions

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates
Differentiation – Discrete Functions

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Forward Difference Approximation

\[ f''(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \]

For a finite 'Δx'

\[ f''(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x} \]
Graphical Representation Of Forward Difference Approximation

**Figure 1** Graphical Representation of forward difference approximation of first derivative.
Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1  Velocity as a function of time

<table>
<thead>
<tr>
<th>t (s)</th>
<th>v(t) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>227.04</td>
</tr>
<tr>
<td>15</td>
<td>362.78</td>
</tr>
<tr>
<td>20</td>
<td>517.35</td>
</tr>
<tr>
<td>22.5</td>
<td>602.97</td>
</tr>
<tr>
<td>30</td>
<td>901.67</td>
</tr>
</tbody>
</table>

Using forward divided difference, find the acceleration of the rocket at $t = 16$ s.
Example 1 Cont.

Solution
To find the acceleration at $t = 16\text{s}$, we need to choose the two values closest to $t = 16\text{s}$, that also bracket $t = 16\text{s}$ to evaluate it. The two points are $t = 15\text{s}$ and $t = 20\text{s}$.

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$t_i = 15$

$t_{i+1} = 20$

$\Delta t = t_{i+1} - t_i$

$= 20 - 15$

$= 5$
Example 1 Cont.

\[ a(16) \approx \frac{v(20) - v(15)}{5} \]

\[ \approx \frac{517.35 - 362.78}{5} \]

\[ \approx 30.914 \text{ m/s}^2 \]
Direct Fit Polynomials

In this method, given \( n + 1 \) data points \((x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

one can fit a \( n^{th} \) order polynomial given by

\[
P_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n
\]

To find the first derivative,

\[
P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \ldots + (n-1)a_{n-1} x^{n-2} + na_n x^{n-1}
\]

Similarly other derivatives can be found.
Example 2-Direct Fit Polynomials

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time

<table>
<thead>
<tr>
<th>t (s)</th>
<th>v(t) (m/s)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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</table>

Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16$ s.
Example 2-Direct Fit Polynomials cont.

Solution

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

\[ v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]

Since we want to find the velocity at \( t = 16 \text{ s} \), and we are using third order polynomial, we need to choose the four points closest to \( t = 16 \text{ s} \) and that also bracket \( t = 16 \text{ s} \) to evaluate it.

The four points are \( t_0 = 10, t_1 = 15, t_2 = 20, \) and \( t_3 = 22.5 \).

- \( t_0 = 10, \ v(t_0) = 227.04 \)
- \( t_1 = 15, \ v(t_1) = 362.78 \)
- \( t_2 = 20, \ v(t_2) = 517.35 \)
- \( t_3 = 22.5, \ v(t_3) = 602.97 \)
Example 2-Direct Fit Polynomials cont.

such that

\[ \nu(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3 \]

\[ \nu(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3 \]

\[ \nu(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3 \]

\[ \nu(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3 \]

Writing the four equations in matrix form, we have

\[
\begin{bmatrix}
1 & 10 & 100 & 1000 \\
1 & 15 & 225 & 3375 \\
1 & 20 & 400 & 8000 \\
1 & 22.5 & 506.25 & 11391
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
227.04 \\
362.78 \\
517.35 \\
602.97
\end{bmatrix}
\]
Example 2-Direct Fit Polynomials cont.

Solving the above four equations gives

\[ a_0 = -4.3810 \]
\[ a_1 = 21.289 \]
\[ a_2 = 0.13065 \]
\[ a_3 = 0.0054606 \]

Hence

\[ v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]
\[ = -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, \quad 10 \leq t \leq 22.5 \]
Example 2-Direct Fit Polynomials cont.

Figure 1 Graph of upward velocity of the rocket vs. time.
Example 2-Direct Fit Polynomials cont.

The acceleration at t=16 is given by

\[ a(16) = \frac{d}{dt} v(t) \bigg|_{t=16} \]

Given that

\[ v(t) = -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, 10 \leq t \leq 22.5 \]

\[ a(t) = \frac{d}{dt} v(t) = \frac{d}{dt} \left( -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3 \right) \]

\[ = 21.289 + 0.26130t + 0.016382t^2, \quad 10 \leq t \leq 22.5 \]

\[ a(16) = 21.289 + 0.26130(16) + 0.016382(16)^2 \]

\[ = 29.664 \text{ m/s}^2 \]
Lagrange Polynomial

In this method, given \((x_1, y_1), \ldots, (x_n, y_n)\), one can fit a \((n-1)^{th}\) order Lagrangian polynomial given by

\[
f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)
\]

where ‘\(n\)’ in \(f_n(x)\) stands for the \(n^{th}\) order polynomial that approximates the function \(y = f(x)\) given at \((n + 1)\) data points as \((x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)\), and

\[
L_i(x) = \prod_{\substack{j=0 \atop j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}
\]

\(L_i(x)\) a weighting function that includes a product of \((n - 1)\) terms with terms of \(j = i\) omitted.
Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating equation (2) gives
Differentiating again would give the second derivative as

\[ f''_2(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f'(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f'(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f'(x_2) \]
Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

**Table 3** Velocity as a function of time

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Determine the value of the acceleration at $t = 16$ s using the second order Lagrangian polynomial interpolation for velocity.
Example 3 Cont.

Solution

\[ v(t) = \left( \frac{t-t_1}{t_0-t_1} \right) \left( \frac{t-t_2}{t_0-t_2} \right) v(t_0) + \left( \frac{t-t_0}{t_1-t_0} \right) \left( \frac{t-t_2}{t_1-t_2} \right) v(t_1) + \left( \frac{t-t_0}{t_2-t_0} \right) \left( \frac{t-t_1}{t_2-t_1} \right) v(t_2) \]

\[ a(t) = \frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)} v(t_0) + \frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)} v(t_1) + \frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)} v(t_2) \]

\[ a(16) = \frac{2(16)-(15+20)}{(10-15)(10-20)} (227.04) + \frac{2(16)-(10+20)}{(15-10)(15-20)} (362.78) + \frac{2(16)-(10+15)}{(20-10)(20-15)} (517.35) \]

\[ = -0.06(227.04) - 0.08(362.78) + 0.14(517.35) \]

\[ = 29.784 \text{m/s}^2 \]
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/discrete_02dif.html