Chapter 03.02
Solution of Cubic Equations

After reading this chapter, you should be able to:

1. find the exact solution of a general cubic equation.

How to Find the Exact Solution of a General Cubic Equation

In this chapter, we are going to find the exact solution of a general cubic equation

\[ ax^3 + bx^2 + cx + d = 0 \]  

(1)

To find the roots of Equation (1), we first get rid of the quadratic term \((x^2)\) by making the substitution

\[ x = y - \frac{b}{3a} \]  

(2)

to obtain

\[ a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0 \]  

(3)

Expanding Equation (3) and simplifying, we obtain the following equation

\[ ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0 \]  

(4)

Equation (4) is called the depressed cubic since the quadratic term is absent. Having the equation in this form makes it easier to solve for the roots of the cubic equation (Click here to know the history behind solving cubic equations exactly).

First, convert the depressed cubic Equation (4) into the form

\[ y^3 + \frac{1}{a}\left(c - \frac{b^2}{3a}\right)y + \frac{1}{a}\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0 \]

\[ y^3 + ey + f = 0 \]  

(5)

where

\[ e = \frac{1}{a}\left(c - \frac{b^2}{3a}\right) \]
Now, reduce the above equation using Vieta’s substitution

\[ y = z + \frac{s}{z} \]  

(6)

For the time being, the constant \( s \) is undefined. Substituting into the depressed cubic Equation (5), we get

\[ \left( z + \frac{s}{z} \right)^3 + e \left( z + \frac{s}{z} \right) + f = 0 \]  

(7)

Expanding out and multiplying both sides by \( z^3 \), we get

\[ z^6 + (3s + e)z^4 + fz^3 + s(3s + e)z^2 + s^3 = 0 \]  

(8)

Now, let \( s = -\frac{e}{3} \) (\( s \) is no longer undefined) to simplify the equation into a tri-quadratic equation.

\[ z^6 + fz^3 - \frac{e^3}{27} = 0 \]  

(9)

By making one more substitution, \( w = z^3 \), we now have a general quadratic equation which can be solved using the quadratic formula.

\[ w^2 + fw - \frac{e^3}{27} = 0 \]  

(10)

Once you obtain the solution to this quadratic equation, back substitute using the previous substitutions to obtain the roots to the general cubic equation.

\[ w \rightarrow z \rightarrow y \rightarrow x \]

where we assumed

\[ w = z^3 \]  

(11)

\[ y = z + \frac{s}{z} \]

\[ s = -\frac{e}{3} \]  

(12)

\[ x = y - \frac{b}{3a} \]

Note: You will get two roots for \( w \) as Equation (10) is a quadratic equation. Using Equation (11) would then give you three roots for each of the two roots of \( w \), hence giving you six root values for \( z \). But the six root values of \( z \) would give you six values of \( y \) (Equation (6)); but three values of \( y \) will be identical to the other three. So one gets only three values of \( y \), and hence three values of \( x \). (Equation (2))

**Example 1**

Find the roots of the following cubic equation.

\[ x^3 - 9x^2 + 36x - 80 = 0 \]
Solution

For the general form given by Equation (1)

\[ ax^3 + bx^2 + cx + d = 0 \]

we have

\[ a = 1, \ b = -9, \ c = 36, \ d = -80 \]

in

\[ x^3 - 9x^2 + 36x - 80 = 0 \]  \hspace{1cm} (E1-1)

Equation (E1-1) is reduced to

\[ y^3 + ey + f = 0 \]

where

\[ e = \frac{1}{a} \left( c - \frac{b^2}{3a} \right) \]
\[ = \frac{1}{1} \left( 36 - \frac{(-9)^2}{3(1)} \right) \]
\[ = 9 \]

and

\[ f = \frac{1}{a} \left( d + \frac{2b^3}{27a^2} - \frac{bc}{3a} \right) \]
\[ = \frac{1}{1} \left( -80 + \frac{2(-9)^3}{27(1)^2} - \frac{(-9)(36)}{3(1)} \right) \]
\[ = -26 \]

giving

\[ y^3 + 9y - 26 = 0 \]  \hspace{1cm} (E1-2)

For the general form given by Equation (5)

\[ y^3 + ey + f = 0 \]

we have

\[ e = 9, \ f = -26 \]

in Equation (E1-2).

From Equation (12)

\[ s = \frac{-e}{3} \]
\[ = \frac{-9}{3} \]
\[ = -3 \]

From Equation (10)

\[ w^2 + fw - \frac{e^3}{27} = 0 \]
\[ w^2 - 26w - \frac{9^3}{27} = 0 \]
\[ w^2 - 26w - 27 = 0 \]
where

\[ w = z^3 \]

and

\[ y = z + \frac{s}{z} \]
\[ = z - \frac{3}{z} \]

\[ w = \frac{-(26) \pm \sqrt{(-26)^2 - 4(1)(-27)}}{2(1)} \]
\[ = 27, -1 \]

The solution is

\[ w_1 = 27 \]
\[ w_2 = -1 \]

Since

\[ w = z^3 \]
\[ z^3 = w \]

For \( w = w_1 \)

\[ z^3 = w_1 = 27 = 27e^{i\theta} \]

Since

\[ w = z^3 \]
\[ re^{i\theta} = (ue^{i\alpha})^3 = u^3 e^{3i\alpha} \]
\[ r(\cos \theta + i \sin \theta) = u^3(\cos 3\alpha + i \sin 3\alpha) \]

resulting in

\[ r = u^3 \]
\[ \cos \theta = \cos 3\alpha \]
\[ \sin \theta = \sin 3\alpha \]

Since \( \sin \theta \) and \( \cos \theta \) are periodic of \( 2\pi \),

\[ 3\alpha = \theta + 2\pi k \]
\[ \alpha = \frac{\theta + 2\pi k}{3} \]

\( k \) will take the value of 0, 1 and 2 before repeating the same values of \( \alpha \).

So,

\[ \alpha = \frac{\theta + 2\pi k}{3}, k = 0, 1, 2 \]
\[ \alpha_1 = \frac{\theta}{3} \]
\[ \alpha_2 = \frac{(\theta + 2\pi)}{3} \]
\[
\alpha_3 = \frac{(\theta + 4\pi)}{3}
\]

So roots of \( w = z^3 \) are
\[
z_1 = r_3^\frac{1}{3} \left( \cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right)
\]
\[
z_2 = r_3^\frac{1}{3} \left( \cos \frac{\theta + 2\pi}{3} + i \sin \frac{\theta + 2\pi}{3} \right)
\]
\[
z_3 = r_3^\frac{1}{3} \left( \cos \frac{\theta + 4\pi}{3} + i \sin \frac{\theta + 4\pi}{3} \right)
\]
gives
\[
z_1 = (27)^{\frac{1}{3}} \left( \cos \frac{0}{3} + i \sin \frac{0}{3} \right)
\]
\[
= 3
\]
\[
z_2 = (27)^{\frac{1}{3}} \left( \cos \frac{0 + 2\pi}{3} + i \sin \frac{0 + 2\pi}{3} \right)
\]
\[
= 3 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)
\]
\[
= 3 \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)
\]
\[
= -\frac{3}{2} + i \frac{3\sqrt{3}}{2}
\]
\[
z_3 = (27)^{\frac{1}{3}} \left( \cos \frac{0 + 4\pi}{3} + i \sin \frac{0 + 4\pi}{3} \right)
\]
\[
= 3 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)
\]
\[
= 3 \left( -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right)
\]
\[
= -\frac{3}{2} - i \frac{3\sqrt{3}}{2}
\]

Since
\[
y = z - \frac{3}{z}
\]
\[
y_1 = z_1 - \frac{3}{z_1}
\]
\[
= 3 - \frac{3}{3}
\]
\[
= 2
\]
\[ y_2 = z_2 - \frac{3}{z_2} \]
\[ = \left( \frac{3}{2} + i \frac{3\sqrt{3}}{2} \right) - \frac{3}{\left( \frac{3}{2} + i \frac{3\sqrt{3}}{2} \right)} \]
\[ = \frac{-5 + 3i \sqrt{3}}{-1 + i \sqrt{3}} \]
\[ = \frac{-5 + 3i \sqrt{3}}{-1 + i \sqrt{3}} \times \frac{-1 - i \sqrt{3}}{-1 - i \sqrt{3}} \]
\[ = 1 + i 2\sqrt{3} \]
\[ y_3 = z_3 - \frac{3}{z_3} \]
\[ = \left( \frac{3}{2} - i \frac{3\sqrt{3}}{2} \right) - \frac{3}{\left( \frac{3}{2} - i \frac{3\sqrt{3}}{2} \right)} \]
\[ = \frac{5 - 3i \sqrt{3}}{1 + i \sqrt{3}} \]
\[ = \frac{5 - 3i \sqrt{3}}{1 + i \sqrt{3}} \times \frac{1 - i \sqrt{3}}{1 - i \sqrt{3}} \]
\[ = 1 - i 2\sqrt{3} \]

Since
\[ x = y + 3 \]
\[ x_1 = y_1 + 3 \]
\[ = 2 + 3 \]
\[ = 5 \]
\[ x_2 = y_2 + 3 \]
\[ = \left( -1 + i 2\sqrt{3} \right) + 3 \]
\[ = 2 + i 2\sqrt{3} \]
\[ x_3 = y_3 + 3 \]
\[ = \left( -1 - i 2\sqrt{3} \right) + 3 \]
\[ = 2 - i 2\sqrt{3} \]

The roots of the original cubic equation
\[ x^3 - 9x^2 + 36x - 80 = 0 \]
are \( x_1, x_2, \) and \( x_3, \) that is,
\[ 5, 2 + i 2\sqrt{3}, 2 - i 2\sqrt{3} \]

Verifying
Solution of Cubic Equations

\[(x - 5)(x - (2 + i2\sqrt{3}))(x - (2 - i2\sqrt{3})) = 0\]
gives
\[x^3 - 9x^2 + 36x - 80 = 0\]
Using
\[w_2 = -1\]
would yield the same values of the three roots of the equation. Try it.

Example 2

Find the roots of the following cubic equation
\[x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0\]

Solution

For the general form
\[ax^3 + bx^2 + cx + d = 0\]
\[a = 1, b = -0.03, c = 0, d = 2.4 \times 10^{-6}\]
Depress the cubic equation by letting (Equation (2))
\[x = y - \frac{b}{3a}\]
\[= y - \frac{(-0.03)}{3(1)}\]
\[= y + 0.01\]
Substituting the above equation into the cubic equation and simplifying, we get
\[y^3 - (3 \times 10^{-4})y + (4 \times 10^{-7}) = 0\]
That gives \(e = -3 \times 10^{-4}\) and \(f = 4 \times 10^{-7}\) for Equation (5), that is, \(y^3 + ey + f = 0\).
Now, solve the depressed cubic equation by using Vieta’s substitution as
\[y = z + \frac{s}{z}\]
to obtain
\[z^6 + (3s - 3 \times 10^{-4})z^4 + (4 \times 10^{-7})z^3 + s(3s - 3 \times 10^{-4})z^2 + s^3 = 0\]
Letting
\[s = -\frac{e}{3} = -\frac{-3 \times 10^{-4}}{3} = 10^{-4}\]
we get the following tri-quadratic equation
\[z^6 + (4 \times 10^{-7})z^3 + 1 \times 10^{-12} = 0\]
Using the following conversion, \(w = z^3\), we get a general quadratic equation
\[w^2 + (4 \times 10^{-7})w + (1 \times 10^{-12}) = 0\]
Using the quadratic equation, the solutions for \(w\) are
\[w = -4 \times 10^{-7} \pm \sqrt{(4 \times 10^{-7})^2 - 4(1\times 10^{-12})}\]
\[2(1)\]
giving
\[ w_1 = -2 \times 10^{-7} + i(9.79795897113 \times 10^{-7}) \]
\[ w_2 = -2 \times 10^{-7} - i(9.79795897113 \times 10^{-7}) \]

Each solution of \( w = z^3 \) yields three values of \( z \). The three values of \( z \) from \( w_1 \) are in rectangular form.

Since \( w = z^3 \),

Then
\[ z = w^{\frac{1}{3}} \]

Let
\[ w = r(\cos \theta + i \sin \theta) = re^{i\theta} \]

then
\[ z = u(\cos \alpha + i \sin \alpha) = ue^{i\alpha} \]

This gives
\[ w = z^3 \]
\[ re^{i\theta} = (ue^{i\alpha})^3 = u^3 e^{3i\alpha} \]
\[ r(\cos \theta + i \sin \theta) = u^3 (\cos 3\alpha + i \sin 3\alpha) \]

resulting in
\[ r = u^3 \]
\[ \cos \theta = \cos 3\alpha \]
\[ \sin \theta = \sin 3\alpha \]

Since \( \sin \theta \) and \( \cos \theta \) are periodic of \( 2\pi \),
\[ 3\alpha = \theta + 2\pi k \]
\[ \alpha = \frac{\theta + 2\pi k}{3} \]

\( k \) will take the value of 0, 1 and 2 before repeating the same values of \( \alpha \).

So,
\[ \alpha = \frac{\theta + 2\pi k}{3}, \quad k = 0, 1, 2 \]
\[ \alpha_1 = \frac{\theta}{3} \]
\[ \alpha_2 = \frac{(\theta + 2\pi)}{3} \]
\[ \alpha_3 = \frac{(\theta + 4\pi)}{3} \]

So the roots of \( w = z^3 \) are
\[ z_1 = r^\frac{1}{3} \left( \cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right) \]
\[ z_2 = r^\frac{1}{3} \left( \cos \frac{\theta + 2\pi}{3} + i \sin \frac{\theta + 2\pi}{3} \right) \]
Solution of Cubic Equations

\[ z_3 = r^3 \left( \cos \frac{\theta + 4\pi}{3} + i \sin \frac{\theta + 4\pi}{3} \right) \]

So for
\[ w_1 = -2 \times 10^{-7} + i (9.79795897113 \times 10^{-7}) \]
\[ r = \sqrt{(-2 \times 10^{-7})^2 + (9.79795897113 \times 10^{-7})^2} \]
\[ = 1 \times 10^{-6} \]
\[ \theta = \tan^{-1} \frac{9.79795897113 \times 10^{-7}}{-2 \times 10^{-7}} \]
\[ = 1.772154248 (2^{nd} \text{ quadrant because } y \text{ (the numerator) is positive and } x \text{ (the denominator) is negative}) \]
\[ z_1 = (1 \times 10^{-6})^\frac{1}{3} \left( \cos \frac{1.772154248}{3} + i \sin \frac{1.772154248}{3} \right) \]
\[ = 0.008305409517 + i0.005569575635 \]
\[ z_2 = (1 \times 10^{-6})^\frac{1}{3} \left( \cos \frac{1.772154248 + 2\pi}{3} + i \sin \frac{1.772154248 + 2\pi}{3} \right) \]
\[ = -0.008976098746 + i0.004407907815 \]
\[ z_3 = (1 \times 10^{-6})^\frac{1}{3} \left( \cos \frac{1.772154248 + 4\pi}{3} + i \sin \frac{1.772154248 + 4\pi}{3} \right) \]
\[ = 0.0006706892313 - i0.009977483448 \]

Compiling
\[ z_1 = 0.008305409518 + i0.005569575634 \]
\[ z_2 = -0.008976098746 + i0.004407907814 \]
\[ z_3 = 6.70689228525 \times 10^{-4} - i0.009977483448 \]

Similarly, the three values of \( z \) from \( w_2 \) in rectangular form are
\[ z_4 = 0.008305409518 - i0.005569575634 \]
\[ z_5 = -0.008976098746 - i0.004407907814 \]
\[ z_6 = 6.70689228525 \times 10^{-4} + i0.009977483448 \]

Using Vieta’s substitution (Equation (6)),
\[ y = z + \frac{s}{z} \]
\[ y = z + \frac{(1 \times 10^{-4})}{z} \]
we back substitute to find three values for \( y \).

For example, choosing
\[ z_1 = 0.008305409518 + i0.005569575634 \]
gives
\[ y_1 = 0.008305409518 + i0.005569575634 + \frac{1 \times 10^{-4}}{0.008305409518 + i0.005569575634} \]
\[
\begin{align*}
&= 0.008305409518 + i0.005569575634 \\
&\quad + \frac{1 \times 10^{-4}}{0.008305409078 + i0.00556957634} \\
&\quad \times \frac{0.008305409518 - i0.00556957634}{0.008305409518 - i0.00556957634} \\
&= 0.008305409518 + i0.005569575634 \\
&\quad + \frac{1 \times 10^{-4}}{1 \times 10^{-4}} (0.008305409518 - i0.00556957634) \\
&= 0.016610819036 \\
\end{align*}
\]

The values of \( z_1, z_2 \) and \( z_3 \) give

\[
\begin{align*}
y_1 &= 0.016610819036 \\
y_2 &= -0.01795219749 \\
y_3 &= 0.001341378457 \\
\end{align*}
\]

respectively. The three other \( z \) values of \( z_4, z_5 \) and \( z_6 \) give the same values as \( y_1, y_2 \) and \( y_3 \), respectively.

Now, using the substitution of

\[
x = y + 0.01
\]

the three roots of the given cubic equation are

\[
\begin{align*}
x_1 &= 0.016610819036 + 0.01 \\
&= 0.026610819036 \\
x_2 &= -0.01795219749 + 0.01 \\
&= -0.00795219749 \\
x_3 &= 0.001341378457 + 0.01 \\
&= 0.011341378457 \\
\end{align*}
\]

**NONLINEAR EQUATIONS**

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