Binary Matrix Operations

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Objectives

1. *add, subtract, and multiply matrices, and*
2. *apply rules of binary operations on matrices.*

Matrix Addition

Two matrices   and can be added only if they are the same size. The addition is then shown as

$$[C]=[A]+[B]$$

where

$$c_{ij}=a_{ij}+b_{ij}$$

Example 1

Add the following two matrices.

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} \qquad [B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

.

Example 1 (cont.)

$$[C] = [A] + [B]$$

$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 5+6 & 2+7 & 3-2 \\ 1+3 & 2+5 & 7+19 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 1 \\ 4 & 7 & 26 \end{bmatrix}$$

Example 2

Blowout r'us store has two store locations and , and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \qquad [B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?

Example 2 (cont.)

$$[C] = [A] + [B]$$

$$= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} + \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} (25+20) & (20+5) & (3+4) & (2+0) \\ (5+3) & (10+6) & (15+15) & (25+21) \\ (6+4) & (16+1) & (7+7) & (27+20) \end{bmatrix}$$

Example 2 (cont.)

The answer then is,

$$= \begin{bmatrix} 45 & 25 & 7 & 2 \\ 8 & 16 & 30 & 46 \\ 10 & 17 & 14 & 47 \end{bmatrix}$$

So if one wants to know the total number of Copper tires sold in quarter 4 at the two locations, we would look at Row 3 – Column 4 to give $c_{34} = 47$.

Matrix Subtraction

Two matrices $[A]$ and $[B]$ can be subtracted only if they are the same size. The subtraction is then given by

$$[D] = [A] - [B]$$

Where

$$d_{ij} = a_{ij} - b_{ij}$$

Example 3

Subtract matrix $[B]$ from matrix $[A]$.

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

Example 3 (cont.)

$$[D] = [A] - [B]$$

$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} (5-6) & (2-7) & (3-(-2)) \\ (1-3) & (2-5) & (7-19) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 & 5 \\ -2 & -3 & -12 \end{bmatrix}$$

Example 4

Blowout r'us store has two store locations and , and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \qquad [B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?

Example 4 (cont.)

$$[D]=[A]-[B]$$

$$=\begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}+\begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

$$=\begin{bmatrix} 25-20 & 20-5 & 3-4 & 2-0 \\ 5-3 & 10-6 & 15-15 & 25-21 \\ 6-4 & 16-1 & 7-7 & 27-20 \end{bmatrix}$$

Example 4 (cont.)

The answer then is,

$$= \begin{bmatrix} 5 & 15 & -1 & 2 \\ 2 & 4 & 0 & 4 \\ 2 & 15 & 0 & 7 \end{bmatrix}$$

So if you want to know how many more Copper tires were sold in quarter 4 in store *A* than store *B*, $d_{34} = 7$. Note that $d_{13} = -1$ implies that store *A* sold 1 less in Michigan tire than store *B* in quarter 3.

Matrix Multiplication

Two matrices [A] and [B] can be multiplied only if the number of columns of [A] is equal to the number of rows of [B] to give

$$[C]_{m\times n} = [A]_{m\times p}[B]_{p\times n}$$

If $[A]$ is a $m\times p$ matrix and $[B]$ is a $p\times n$ matrix, the resulting matrix $[C]$ is a $m\times n$ matrix.

Matrix Multiplication

So how does one calculate the elements of $[C]$ matrix?

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots + a_{ip}b_{pj}$$

for each $i = 1, 2, \ldots\ldots, m$ and $j = 1, 2, \ldots\ldots, n$

To put it in simpler terms, the i^{th} row and j^{th} column of the $[C]$ matrix in $[C] = [A][B]$ is calculated by multiplying the i^{th} row of $[A]$ by the j^{th} column of $[B]$.

Example 5

Given the following two matrices,

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} \qquad [B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix}$$

Find their product,

$$[C] = [A][B]$$

Example 5 (cont.)

c_{12} be found by multiplying the first row of $[A]$ by the second column of $[B]$,

$$c_{12} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$

$$= -56$$

Example 5 (cont.)

Similarly, one can find the other elements of $[C]$ to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}$$

Example 6

Blowout r'us store has two store locations and , and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \qquad [B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4.

Example 6 (cont.)

Find the per quarter sales of store A if the following are the prices of each tire.

Tirestone = $33.25
Michigan = $40.19
Copper = $25.03

The answer is given by multiplying the price matrix by the quantity of sales of store A. The price matrix is $\begin{bmatrix} 33.25 & 40.19 & 25.03 \end{bmatrix}$.

Example 6 (cont.)

Therefore, the per quarter sales of store A dollars is given by the four columns of the row vector

$$[C] = [1182.38 \quad 1467.38 \quad 877.81 \quad 1747.06]$$

Remember since we are multiplying a 1×3 matrix by a 3×4 matrix, the resulting matrix is a 1×4 matrix.

Scalar Product Of a Constant And a Matrix

If $[A]$ is a $n \times n$ matrix and k is a real number, then the scalar product of k and $[A]$ is another $n \times n$ matrix $[B]$, where $b_{ij} = k\,a_{ij}$.

Example 7

Given the matrix,

$$[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

Find $2[A]$

Example (cont.)

The solution to the product of a scalar and a matrix by the following method,

$$2[A] = 2\begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2.1 & 2\times 3 & 2\times 2 \\ 2\times 5 & 2\times 1 & 2\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix}$$

Combining Linear Matrices

If $[A_1],[A_2],.....,[A_p]$ are matrices of the same size and $k_1,k_2,.....,k_p$ are scalars, then $k_1[A_1]+k_2[A_2]+........+k_p[A_p]$ is called a linear combination of

$[\mathrm{A}_1],[A_2],.....,[A_p]$

Example 8

If

$$[A_1] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix}, [A_2] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}, [A_3] = \begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}$$

then find

$$[A_1] + 2[A_2] - 0.5[A_3]$$

Example 8 (cont.)

$$[A_1] + 2[A_2] - 0.5[A_3]$$

$$= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} - 0.5\begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 1.1 & 1 \\ 1.5 & 1.75 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9.2 & 10.9 & 5 \\ 11.5 & 2.25 & 10 \end{bmatrix}$$

Binary Matrix Operations

Commutative law of addition

If $[A]$ and $[B]$ are $m \times n$ matrices, then

$$[A]+[B]=[B]+[A]$$

Associative law of addition

If $[A]$, $[B]$ and $[C]$ are $m \times n$, $n \times p$, and $p \times r$ size matrices, respectively, then

$$[A]+([B]+[\mathrm{C}])=([A]+[B])+[C]$$

Associative law of multiplication

If $[A]$, $[B]$ and $[C]$ are all $m \times n$, $n \times p$ and $p \times r$ size matrices, respectively, then

$$[A]([B][C])=([A][B])[C]$$

and the resulting matrix size on both sides of the equation is $m \times r$.

Binary Matrix Operations

Distributive Law

If $[A]$ and $[B]$ are $m \times n$ matrices, and $[C]$ and $[D]$ are $n \times p$ size matrices

$$[A]([C]+[D])=[A][C]+[A][D]$$

$$([A]+[B])[C]=[A][C]+[B][C]$$

and the resulting matrix size on both sides of the equation is $m \times r$.

Example 9

Illustrate the associative law of multiplication of matrices using

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix}, \quad [B] = \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix}, \quad [C] = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

Example 9 (cont.)

$$[B][C] =$$

$$= \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix}$$

$$[A]([B][C]) = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix}$$

$$= \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix}$$

Example 9 (cont.)

$$[A][B]=\begin{bmatrix}1 & 2\\ 3 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 9 & 6\end{bmatrix}$$

$$=\begin{bmatrix}20 & 17\\ 51 & 45\\ 18 & 12\end{bmatrix}$$

$$([A][B])[C]=\begin{bmatrix}20 & 17\\ 51 & 45\\ 18 & 12\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}$$

$$=\begin{bmatrix}91 & 105\\ 237 & 276\\ 72 & 78\end{bmatrix}$$

Is $[A][B]=[B][A]$?

If $[A][B]$ exists, number of columns of $[A]$ has to be same as the number of rows of $[B]$ and if $[B][A]$ exists, number of columns of $[B]$ has to be same as the number of rows of $[A]$.

Now for $[A][B]=[B][A]$, the resulting matrix from $[A][B]$ and $[B][A]$ has to be of the same size. This is only possible if $[A]$ and [B] are square and are of the same size. Even then in general $[A][B] \neq [B][A]$.

Example 10

Determine if

$$[A][B]=[B][A]$$

for the following matrices

$$[A]=\begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}, \quad [B]=\begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

Example 10 (cont.)

$$[A][B] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 27 \\ -1 & 29 \end{bmatrix}$$

$$[B][A] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 \\ 16 & 28 \end{bmatrix}$$

Therefore

$$[A][B] \neq [B][A]$$