Part: Cholesky and LDL^T Decomposition

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Lecture # 1

Chapter 04.09: Cholesky and LDL^T Decomposition

Major: All Engineering Majors

Authors: Duc Nguyen

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Numerical Methods for STEM undergraduates

Introduction

$$[A][x]=[b] \quad (1)$$

where

$[A]$ = known coefficient matrix, with dimension $n \times n$

$[b]$ = known <u>r</u>ight-<u>h</u>and-<u>s</u>ide (RHS) $n \times 1$ vector

$[x]$ = unknown $n \times 1$ vector.

Symmetrical Positive Definite (SPD) SLE

A matrix $[A]_{n\times n}$ can be considered as SPD if either of the following conditions is satisfied:

(a) If each and every determinant of sub-matrix $A_{ii}(i=1,2,...,n)$ is positive, or..

(b) If $y^T Ay > 0$, for any given vector $[y]_{n\times 1} \neq \vec{0}$

As a quick example, let us make a test a test to see if the given matrix

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$ is SPD?

Symmetrical Positive Definite (SPD) SLE

<u>Based on criteria (a):</u>

The given 3×3 matrixis symmetrical, because

$$a_{ij} = a_{ji}$$

Furthermore,

$$\det\left|[A]_{1\times 1}\right| = |2| = 2 > 0$$

$$\det\left|[A]_{2\times 2}\right| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 3 > 0$$

$$\det\left|[A]_{3\times 3}\right| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 1 > 0$$

Hence is $[A]$ SPD.

Based on criteria (b): For any given vector

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \neq \vec{0}$$

, one computes

$$scalar = y^T A y$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \left(2y_1^2 - 2y_1y_2 + 2y_2^2\right) + \left\{y_3^2 - 2y_2y_3\right\}$$

$$= (y_1 - y_2)^2 + y_1^2 + y_2^2 + \left\{y_3^2 - 2y_2y_3\right\}$$

$$scalar = (y_1 - y_2)^2 + y_1^2 + (y_2 - y_3)^2 > 0$$

hence matrix is $[A]$ SPD

Step 1: Matrix Factorization phase

$$[A]=[U]^T[U] \tag{2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{3}$$

Multiplying two matrices on the right-hand-side (RHS) of Equation (3), one gets the following 6 equations

$$u_{11}=\sqrt{a_{11}} \qquad u_{12}=\frac{a_{12}}{u_{11}} \qquad u_{13}=\frac{a_{13}}{u_{11}} \tag{4}$$

$$u_{22}=\left(a_{22}-u_{12}^2\right)^{\frac{1}{2}} \qquad u_{23}=\frac{a_{23}-u_{12}u_{13}}{u_{22}} \qquad u_{33}=\left(a_{33}-u_{13}^2-u_{23}^2\right)^{\frac{1}{2}} \tag{5}$$

$$u_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} (u_{ki})^2 \right)^{\frac{1}{2}} \qquad (6)$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \qquad (7)$$

Step 1.1: Compute the numerator of Equation (7), such as

$$Sum = a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}$$

Step 1.2 If u_{ij} is an off-diagonal term (say $i < j$) then $u_{ij} = \frac{Sum}{u_{ii}}$ (See Equation (7)). Else, if u_{ij} is a diagonal term (that is, $i = j$), then $u_{ii} = \sqrt{Sum}$ (See Equation (6))

As a quick example, one computes:

$$u_{57} = \frac{a_{57} - u_{15}u_{17} - u_{25}u_{27} - u_{35}u_{37} - u_{45}u_{47}}{u_{55}} \qquad (8)$$

Thus, for computing $u(i = 5, j = 7)$, one only needs to use the (already factorized) data in columns $\# i(= 5)$ and $\# j(= 7)$ of $[U]$, respectively.

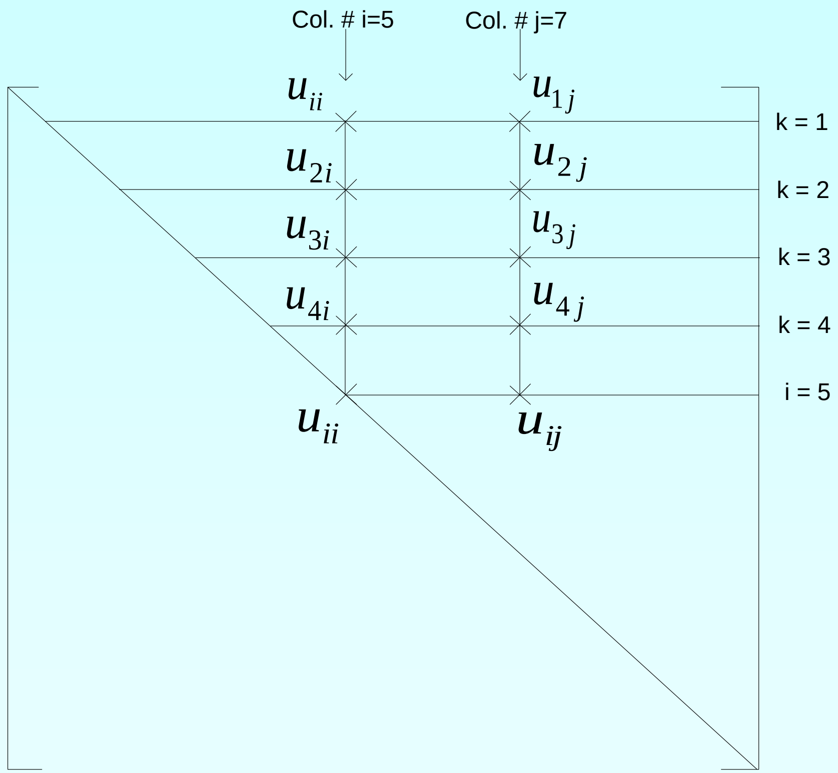

Figure 1: Cholesky Factorization for the term u_{ij}

Step 2: Forward Solution phase

Substituting Equation (2) into Equation (1), one gets:

$$[U]^T[U][x]=[b] \quad (9)$$

Let us define:

$$[U][x]\equiv[y] \quad (10)$$

Then, Equation (9) becomes:

$$[U]^T[y]=[b] \quad (11)$$

$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \quad (12)$$

$$u_{11}y_1 = b_1$$

$$y_1 = \frac{b_1}{u_{11}} \tag{13}$$

From the 2nd row of Equation (12), one gets

$$u_{12}y_1 + u_{22}y_2 = b_2$$

$$y_2 = b_2 - \frac{u_{12}y_1}{u_{22}} \tag{14}$$

Similarly

$$y_3 = \frac{b_3 - u_{13}y_1 - u_{23}y_2}{u_{33}} \tag{15}$$

In general, from the j^{th} row of Equation (12), one has

$$y_j = \frac{b_j - \sum_{i=1}^{j-1} u_{ij} y_i}{u_{jj}} \qquad (16)$$

Step 3: Backward Solution phase

As a quick example, one has (See Equation (10)):

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (17)$$

From the last (or $n^{th} = 3^{rd}$) row of Equation (17),

one has $$u_{33}x_3 = y_3$$

hence $$x_3 = \frac{y_3}{u_{33}} \qquad (18)$$

Similarly: $$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}} \qquad (19)$$

and $$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}} \qquad (20)$$

In general, one has:

$$x_j = \frac{y_j - \sum_{i=j+1}^{n} u_{ji} x_i}{u_{jj}} \qquad (21)$$

$$[A]=[L][D][L]^T \tag{22}$$

For example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Multiplying the three matrices on the RHS of Equation (23), one obtains the following formulas for the “diagonal” $[D]$, and “lower-triangular” $[L]$ matrices:

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk} \qquad (24)$$

$$l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk} \right) \times \left(\frac{1}{d_{jj}} \right) \qquad (25)$$

Step1: Factorization phase

$$[A]=[L][D][L]^T \qquad (22,\ \text{repeated})$$

Step 2: Forward solution and diagonal scaling phase

Substituting Equation (22) into Equation (1), one gets:

$$[L][D][L]^T[x]=[b] \qquad (26)$$

Let us define:

$$[L]^T[x]=[y]$$

Also, define: $[D][y]=[z]$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (29)$$

$$y_i=\frac{z_i}{d_{ii}},\ for\ i=1,2,3,\ldots\ldots,n \quad (30)$$

Then Equation (26) becomes:

$$[L][z]=[b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{31}$$

$$z_i = b_i - \sum_{k=1}^{i-1} L_{ik} z_k \quad for\ i = 1,2,3,\ldots\ldots,n \tag{32}$$

Step 3: Backward solution phase

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_i = y_i - \sum_{k=i+1}^{n} l_{ki} x_k ;\ for\ i = n, n-1, \ldots\ldots, 1$$

Numerical Example 1 (Cholesky algorithms)

Solve the following SLE system for the unknown vector $[x]$?

$$[A][x]=[b]$$

where

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[b]=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

The factorized, $[U]$ upper triangular matrix can be computed by either referring to Equations (6-7), or looking at Figure 1, as following:

$$\left.\begin{aligned}
u_{11} &= \sqrt{a_{11}} \\
&= \sqrt{2} \\
&= 1.414 \\
u_{12} &= \frac{a_{12}}{u_{11}} \\
&= \frac{-1}{1.414} \\
&= -0.7071 \\
u_{13} &= \frac{a_{13}}{u_{11}} \\
&= \frac{0}{1.414} \\
&= 0
\end{aligned}\right\} row\,1\,of\ [U]$$

$$
\left.
\begin{aligned}
u_{22} &= \left\{ a_{22} - \sum_{k=1}^{i-1=1} \left(u_{ki} \right)^2 \right\}^{\frac{1}{2}} \\
&= \left\{ 2 - \left(u_{12} \right)^2 \right\}^{\frac{1}{2}} \\
&= \sqrt{2 - \left(-0.7071 \right)^2} \\
&= 1.225 \\
u_{23} &= \frac{a_{23} - \sum\limits_{k=1}^{i-1=1} u_{ki} u_{kj}}{U_{22}} \\
&= \frac{-1 - u_{12} \times u_{13}}{1.225} \\
&= \frac{-1 - \left(-0.7071 \right)\left(0 \right)}{1.225} \\
&= -0.8165
\end{aligned}
\right\} row\ 2\ of\ [U]
$$

$$\left.\begin{aligned} u_{33} &= \left\{ a_{33} - \sum_{k=1}^{i-1=2} \left(u_{ki}\right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ a_{33} - u_{13}^2 - u_{23}^2 \right\}^{\frac{1}{2}} \\ &= \sqrt{1 - (0)^2 - (-0.8165)^2} \\ &= 0.5774 \end{aligned}\right\} row\,3\,of\ [U]$$

Thus, the factorized matrix

$$[U] = \begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

The <u>forward solution</u> phase, shown in Equation (11), becomes:

$$[U]^T[y]=[b]$$

$$\begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y_1 = \frac{b_1}{u_{11}}$$

$$= \frac{1}{1.414}$$

$$= 0.7071$$

$$y_2 = \frac{b_2 - \sum_{i=1}^{j-1=1} u_{ij} y_i}{u_{jj}}$$

$$= \frac{0 - (u_{12} = -0.7071)(y_1 = 0.7071)}{(u_{22} = 1.225)}$$

$$= 0.4082$$

$$y_3 = \frac{b_3 - \sum_{i=1}^{j-1=2} u_{ij} y_i}{u_{jj}}$$

$$= \frac{0 - (u_{13} = 0)(y_1 = 0.7071) - (u_{23} = -0.8165)(y_2 = 0.4082)}{(u_{33} = 0.5774)}$$

$$= 0.5774$$

The backward solution phase, shown in Equation (10), becomes:

$$[U][x]=[y]$$

$$\begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

$$x_3 = \frac{y_j}{u_{jj}}$$

$$= \frac{y_3}{u_{33}}$$

$$= \frac{0.5774}{0.5774}$$

$$= 1$$

$$x_2 = \frac{y_j - \sum_{i=j+1=3}^{N=3} u_{ji} x_i}{u_{jj}}$$

$$= \frac{y_2 - u_{23} x_3}{u_{22}}$$

$$= \frac{0.4082 - (-0.8165)(1)}{1.225}$$

$$= 1$$

$$x_1 = \frac{y_j - \sum_{i=j+1=2}^{N=3} u_{ji} x_i}{u_{jj}}$$

$$= \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}$$

$$= \frac{0.7071 - (-0.7071)(1) - (0)(1)}{1.414}$$

$$= 1$$

Hence

$$[x] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Numerical Example 2

(LDL^T Algorithms)

Using the same data given in Numerical Example 1, find the unknown vector $[x]$ by LDL^T algorithms?

Solution:

The factorized matrices $[D]$ and $[L]$ can be computed

from Equation (24), and Equation (25), respectively.

$$
\left.
\begin{aligned}
d_{11} &= a_{11} - \sum_{k=1}^{j-1=0} l_{jk}^2 d_{kk} \\
&= a_{11} \\
&= 2 \\
l_{11} &= 1\,(always\,!) \\
l_{21} &= \frac{a_{21} - \sum_{k=1}^{j-1=0} l_{ik} d_{kk} l_{jk}}{d_{jj}} \\
&= \frac{a_{21}}{d_{11}} \\
&= \frac{-1}{2} \\
&= -0.5 \\
l_{31} &= \frac{a_{31}}{d_{11}} \\
&= \frac{0}{2} \\
&= 0
\end{aligned}
\right\}
Column\,1\,of\ \ matrices\,of\ \left[D\right] and \left[L\right]
$$

$$\left.\begin{aligned}
d_{22} &= a_{22} - \sum_{k=1}^{j-1=1} l_{jk}^2 d_{kk} \\
&= 2 - l_{21}^2 d_{11} \\
&= 2 - (-0.5)^2(2) \\
&= 1.5 \\
l_{22} &= 1\,(always\,!) \\
l_{32} &= \frac{a_{32} - \sum_{k=1}^{j-1=1} l_{31} d_{11} l_{21}}{d_{22}} \\
&= \frac{-1-(0)(2)(-0.5)}{1.5} \\
&= -0.6667
\end{aligned}\right\} Column\ 2\ of\ \ matrices\ [D]\ and\ [L]$$

$$
\left.\begin{aligned}
d_{33} &= a_{33} - \sum_{k=1}^{j-1=2} l_{jk}^2 d_{kk} \\
&= 1 - l_{31}^2 d_{11} - l_{32}^2 d_{22} \\
&= 1 - (0)^2(2) - (-0.6667)^2(1.5) \\
&= 0.3333
\end{aligned}\right\} Column\ 3\ of\ matrices\ [D]\ and\ [L]
$$

Hence

$$[D]=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

and

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix}$$

The <u>forward solution</u> shown in Equation (31), becomes:

$$[L][z]=[b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.667 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

or,

$$z_i = b_i - \sum_{k=1}^{i-1} l_{ik} z_k \qquad \text{(32, repeated)}$$

Hence

$$z_1 = b_1 = 1$$

$$\begin{aligned} z_2 &= b_2 - L_{21} z_1 \\ &= 0 - (-0.5)(1) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} z_3 &= b_3 - L_{31} z_1 - L_{32} z_2 \\ &= 0 - (0)(1) - (-0.6667)(0.5) \\ &= 0.3333 \end{aligned}$$

The diagonal scaling phase, shown in Equation (29), becomes

$$[D][y]=[z]$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.3333 \end{bmatrix}$$

or

$$y_i = \frac{z_i}{d_{ii}}$$

Hence

$$y_1 = \frac{z_1}{d_{11}} \quad = \frac{1}{2} \quad = 0.5$$

$$y_2 = \frac{z_2}{d_{22}} \quad = \frac{0.5}{1.5} \quad = 0.3333$$

$$y_3 = \frac{z_3}{d_{33}} \quad = \frac{0.3333}{0.3333} \quad = 1$$

The <u>backward solution</u> phase can be found by referring to Equation (27)

$$[L]^T[x]=[y]$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.667 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

$$x_i = y_i - \sum_{k=i+1}^{N} l_{ki} x_k \qquad \text{(28, repeated)}$$

Hence

$$x_3 = y_3$$
$$= 1$$

$$x_2 = y_2 - l_{32}x_3$$
$$= 0.3333 - (-0.6667) \times 1$$
$$x_2 = 1$$

$$x_1 = y_1 - l_{21}x_2 - l_{31}x_3$$
$$x_1 = 0.5 - (-0.5)(1) - (0)(1)$$
$$= 1$$

Hence

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Re-ordering Algorithms For Minimizing Fill-in Terms [1,2].

During the factorization phase (of Cholesky, or LDL^T Algorithms), many "zero" terms in the original/given $[A]$ matrix will become "non-zero" terms in the factored matrix $[U]$. These new non-zero terms are often called as "fill-in" terms (indicated by the symbol F) It is, therefore, highly desirable to minimize these fill-in terms , so that both computational time/effort and computer memory requirements can be substantially reduced.

For example, the following matrix $[A]$ and vector $[b]$ are given:

$$[A] = \begin{bmatrix} 112 & 7 & 0 & 0 & 0 & 2 \\ 7 & 110 & 5 & 4 & 3 & 0 \\ 0 & 5 & 88 & 0 & 0 & 1 \\ 0 & 4 & 0 & 66 & 0 & 0 \\ 0 & 3 & 0 & 0 & 44 & 0 \\ 2 & 0 & 1 & 0 & 0 & 11 \end{bmatrix} \quad (33)$$

$$[b] = \begin{bmatrix} 121 \\ 129 \\ 94 \\ 70 \\ 47 \\ 14 \end{bmatrix} \quad (34)$$

The Cholesky factorization matrix $[U]$, based on the original matrix $[A]$ (see Equation 33) and Equations (6-7), or Figure 1, can be symbolically computed as:

$$[U]=\begin{bmatrix} \times & \times & 0 & 0 & 0 & \times \\ 0 & \times & \times & \times & \times & F \\ 0 & 0 & \times & F & F & \times \\ 0 & 0 & 0 & \times & F & F \\ 0 & 0 & 0 & 0 & \times & F \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \quad (35)$$

IPERM (<u>new</u> equation #) = {<u>old</u> equation #} (36)

such as, for this particular example:

$$IPERM\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}=\begin{bmatrix}6\\5\\4\\3\\2\\1\end{bmatrix} \qquad (37)$$

Using the above results (see Equation 37), one will be able to construct the following re-arranged matrices:

$$[A^*] = \begin{bmatrix} 11 & 0 & 0 & 1 & 0 & 2 \\ 0 & 44 & 0 & 0 & 3 & 0 \\ 0 & 0 & 66 & 0 & 4 & 0 \\ 1 & 0 & 0 & 88 & 5 & 0 \\ 0 & 3 & 4 & 5 & 110 & 7 \\ 2 & 0 & 0 & 0 & 7 & 112 \end{bmatrix} \quad (38)$$

and

$$[b^*] = \begin{bmatrix} 14 \\ 47 \\ 70 \\ 94 \\ 129 \\ 121 \end{bmatrix} \quad (39)$$

Remarks:

In the original matrix A (shown in Equation 33), the nonzero term A^{*} (old row 1, old column 2) = 7 will move to new location of the new matrix A^{*} (new row 6, new column 5) = 7, etc.

The non zero term A (old row 3, old column 3) = 88 will move to A^{*} (new row 4, new column 4) = 88, etc.

The value of b (old row 4) = 70 will be moved to (or located at) b^{*} (new row 3) = 70, etc

Now, one would like to solve the following modified system of linear equations (SLE) for $[x^*]$,

$$[A^*][x^*]=[b^*] \qquad (40)$$

rather than to solve the original SLE (see Equation1). The original unknown vector $\{x\}$ can be easily recovered from $[x^*]$ and $[IPERM]$,shown in Equation (37).

The factorized matrix $[U^{*}]$ can be "symbolically" computed from $[A^{*}]$ as (by referring to either Figure 1, or Equations 6-7):

$$[U^{*}]=\begin{bmatrix} \times & 0 & 0 & \times & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & \times & 0 \\ 0 & 0 & 0 & \times & \times & F \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \tag{41}$$

4. On-Line Chess-Like Game For Reordering/Factorized Phase [4].

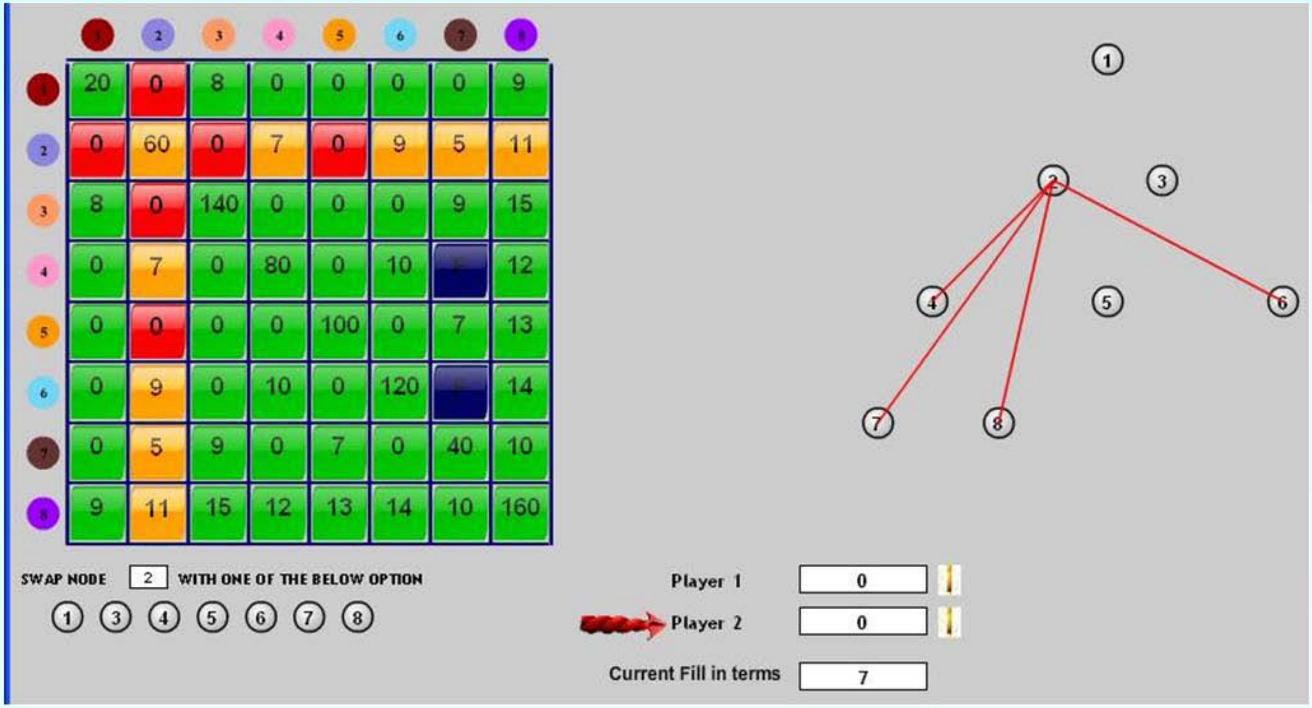

Figure 2 A Chess-Like Game For Learning to Solve SLE.

(A)Teaching undergraduate/HS students the process how to use the reordering output IPERM(-), see Equations (36-37) for converting the original/given matrix $[A]$, see Equation (33), into the new/modified matrix $[A^*]$, see Equation (38). This step is reflected in Figure 2, when the “Game Player” decides to swap node (or equation) $"i"$ (say $i=2$) with another node (or equation $"j"$) , and click the “CONFIRM” icon!

Since node $"i=2"$ is currently connected to nodes $j=4,6,7,8;$ hence swapping node $i=2$ with the above nodes $"j"$ will “NOT” change the number/pattern of “Fill-in” terms. However, if node $i=2$ is swapped with node $j=1, or\ 3, or\ 5,$ then the fill-in terms pattern may change (for better or worse)!

(B) Helping undergraduate/HS students to understand the “symbolic” factorization” phase, by symbolically utilizing the Cholesky factorized Equations (6-7). This step is illustrated in Figure 2, for which the “game player” will see (and also hear the computer animated sound, and human voice), the non-zero terms (including fill-in terms) of the original matrix to move to the new locations in the new/modified $[A]$ matrix $[A^*]$.

(C) Helping undergraduate/HS students to understand the “numerical factorization” phase, by numerically utilizing the same Cholesky factorized Equations (6-7).

(D) Teaching undergraduate engineering/science students and even high-school (HS) students to “understand existing reordering concepts”, or even to “discover new reordering algorithms”

5. Further Explanation On The Developed Game

1. In the above Chess-Like Game, which is available on-line [4], powerful features of FLASH computer environments [3], such as animated sound, human voice, motions, graphical colors etc... have all been incorporated and programmed into the developed game-software for more appealing to game players/learners.

2. In the developed “Chess-Like Game”, fictitious monetary (or any kind of ‘scoring system”) is rewarded (and broadcasted by computer animated human voice) to game players, based on how he/she swaps the node (or equation) numbers, and consequently based on how many fill-in terms occurred. In general, less fill-in "F" terms introduced will result in more rewards!

3. Based on the original/given matrix $[A]$, and existing re-ordering algorithms (such as the Reverse Cuthill-Mckee, or RCM algorithms [1-2]) the number of fill-in terms $("F")$ can be computed (using RCM algorithms). This internally generated information will be used to judge how good the players/learners are, and/or broadcast “congratulations message” to a particular player who discovers new “chess-like move” (or, swapping node) strategies which are even better than RCM algorithms!

4. Initially, the player(s) will select the matrix size (8×8 , or larger is recommended), and the percentage (50%, or larger is suggested) of zero-terms (or sparsity of the matrix). Then, “START Game” icon will be clicked by the player.

5. The player will then CLICK one of the selected node "i" (or equation) numbers appearing on the computer screen. The player will see those nodes "j" which are connected to node "i" (based on the given/generated matrix $[A]$). The player then has to decide to swap node "i" with one of the possible node "j"

After confirming the player's decision, the outcomes/ results will be announced by the computer animated human voice, and the monetary-award will (or will NOT) be given to the players/learners, accordingly. In this software, a maximum of $1,000,000 can be earned by the player, and the "exact dollar amount" will be INVERSELY proportional to the number of fill-in terms occurred (as a consequence of the player's decision on how to swap node "i" with another node "j").

6. The next player will continue to play, with his/her move (meaning to swap the i^{th} node with the j^{th} node) based on the current best non-zero terms pattern of the matrix.

THE END

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