Introduction

Autar Kaw
Benjamin Rigsby

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Transforming Numerical Methods Education for STEM Undergraduates

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Objectives

1. *define what a matrix is*
2. *identify special types of matrices, and*
3. *identify when two matrices are equal.*

What does a matrix look like?

Matrices are everywhere. If you have used a spreadsheet such as Excel or Lotus or written a table, you have used a matrix. Matrices make presentation of numbers clearer and make calculations easier to program.

Look at the matrix below about the sale of tires in a Blowoutr'us store – given by quarter and make of tires.

$$\begin{array}{ll} & \begin{array}{cccc} Q1 & Q2 & Q3 & Q4 \end{array} \\ \begin{array}{l} \textbf{Tirestone} \\ \textbf{Michigan} \\ \textbf{Copper} \end{array} & \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \end{array}$$

If one wants to know how many *Copper* tires were sold in *Quarter 4*, we go along the row *Copper* and column *Q4* and find that it is 27.

What is a matrix?

A *matrix* is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix $[A]$ is denoted by

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$$

What is a matrix?

Row i of $[A]$ has n elements and is

$$\begin{bmatrix} a_{i1} & a_{i2} \ldots . a_{in} \end{bmatrix}$$

and column j of $[A]$ has m elements and is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

What is a matrix?

Each matrix has rows and columns and this defines the size of the matrix. If a matrix [A] has m rows and n columns, the size of the matrix is denoted by $m \times n$. The matrix [A] may also be denoted by $[A]_{mxn}$ to show that [A] is a matrix with m rows and n columns.

Each entry in the matrix is called the entry or element of the matrix and is denoted by a_{ij} where I is the row number and j is the column number of the element.

What is a matrix?

The matrix for the tire sales example could be denoted by the matrix [A] as

$$[A]=\begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

There are 3 rows and 4 columns, so the size of the matrix is 3×4. In the above [A] matrix, a_{34} =27.

Special Types of Matrices

- Row Vector
- Column Vector
- Submatrix
- Square Matrix
- Upper Triangular Matrix
- Lower Triangular Matrix
- Diagonal Matrix
- Identity Matrix
- Zero Matrix
- Tri-diagonal Matrices
- Diagonally Dominant Matrix

What Is a Vector?

What is a vector?

A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

Row Vector:

If a matrix $[B]$ has one row, it is called a row vector $[B] = [b_1\ b_2\ \ldots\ldots b_n]$ and n is the dimension of the row vector.

Column vector:

If a matrix $[C]$ has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

and m is the dimension of the vector.

Row Vector

Example 1

An example of a row vector is as follows,

$$[B] = [25 \;\; 20 \;\; 3 \;\; 2 \;\; 0]$$

$[B]$ is an example of a row vector of dimension 5.

Column Vector

Example 2

An example of a column vector is as follows,

$$[C] = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}$$

$[C]$ is an example of a row vector of dimension 5.

Submatrix

If some row(s) or/and column(s) of a matrix $[A]$ are deleted (no rows or columns may be deleted), the remaining matrix is called a submatrix of $[A]$.

Example 3

Find some of the submatrices of the matrix

$$[A] = \begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

Square Matrix

If the number of rows m a matrix is equal to the number of columns n of a matrix [A], ($m=n$), then [A] is called a square matrix. The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the *diagonal elements* of a square matrix. Sometimes the diagonal of the matrix is also called the *principal or main of the matrix*.

Example 4

Give an example of a square matrix.

$$[A] = \begin{bmatrix} 25 & 20 & 3 \\ 5 & 10 & 15 \\ 6 & 15 & 7 \end{bmatrix}$$

is a square matrix as it has the same number of rows and columns, that is, 3.

The diagonal elements of $[A]$ are $a_{11} = 25,\ a_{22} = 10,\ a_{33} = 7$.

Upper Triangular Matrix

A $m \times n$ matrix for which $a_{ij} = 0, i > j$ is called an upper triangular matrix. That is, all the elements below the diagonal entries are zero.

Example 5

Give an example of an upper triangular matrix.

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix}$$

is an upper triangular matrix.

Lower Triangular Matrix

A $m \times n$ matrix for which $a_{ij} = 0,\ j > i$ is called an lower triangular matrix. That is, all the elements above the diagonal entries are zero.

Example 6

Give an example of a lower triangular matrix.

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.6 & 2.5 & 1 \end{bmatrix}$$

is a lower triangular matrix.

Diagonal Matrix

A square matrix with all non-diagonal elements equal to zero is called a diagonal matrix, that is, only the diagonal entries of the square matrix can be non-zero, $(a_{ij} = 0,\ i \neq j)$.

Example 7

An example of a diagonal matrix.

$$[A]=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any or all the diagonal entries of a diagonal matrix can be zero.

$$[A]=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is also a diagonal matrix.

Identity Matrix

A diagonal matrix with all diagonal elements equal to one is called an identity matrix, ($a_{ij} = 0, \; i \neq j$ and $a_{ii} = 1$ for all i).

An example of an identity matrix is,

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Zero Matrix

A matrix whose all entries are zero is called a zero matrix, ($a_{ij} = 0$ for all i and j).

Some examples of zero matrices are,

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\mathrm{B}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Tridiagonal Matrix

A tridiagonal matrix is a square matrix in which all elements not on the following are zero - the major diagonal, the diagonal above the major diagonal, and the diagonal below the major diagonal.

An example of a tridiagonal matrix is,

$$[A] = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 2 & 3 & 9 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

Non-square Matrix

Do non-square matrices have diagonal entries?

Yes, for a $m \times n$ matrix $[A]$, the diagonal entries are $a_{11}, a_{22} \ldots, a_{k-1,k-1}, a_{kk}$ where $k = \min\{m,n\}$.

Example 11

What are the diagonal entries of

$$[A]=\begin{bmatrix}3.2 & 5\\ 6 & 7\\ 2.9 & 3.2\\ 5.6 & 7.8\end{bmatrix}$$

The diagonal elements of [A] are $a_{11}=3.2$ and $a_{22}=7$.

Diagonally Dominant Matrix

A $n \times n$ square matrix $[A]$ is a diagonally dominant matrix if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \quad \text{for all } i = 1,2,\ldots\ldots,n \quad \text{and}$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \quad \text{for at least one } i,$$

that is, for each row, the absolute value of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row, and that the inequality is strictly greater than for at least one row. Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear equations.

Example 12

Give examples of diagonally dominant matrices and not diagonally dominant matrices.

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

is a diagonally dominant matrix as

$$|a_{11}| = |15| = 15 \geq |a_{12}| + |a_{13}| = |6| + |7| = 13$$

$$|a_{22}| = |-4| = 4 \geq |a_{21}| + |a_{23}| = |2| + |-2| = 4$$

$$|a_{33}| = |6| = 6 \geq |a_{31}| + |a_{32}| = |3| + |2| = 5$$

and for at least one row, that is Rows 1 and 3 in this case, the inequality is a strictly greater than inequality.

Example 12 (cont.)

$$[B] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4 & 2 \\ 3 & -2 & 5.001 \end{bmatrix}$$

is a diagonally dominant matrix as

$$|b_{11}| = |-15| = 15 \geq |b_{12}| + |b_{13}| = |6| + |9| = 15$$

$$|b_{22}| = |-4| = 4 \geq |b_{21}| + |b_{23}| = |2| + |2| = 4$$

$$|b_{33}| = |5.001| = 5.001 \geq |b_{31}| + |b_{32}| = |3| + |-2| = 5$$

The inequalities are satisfied for all rows and it is satisfied strictly greater than for at least one row (in this case it is Row 3).

Example 12 (cont.)

$$[C] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is not diagonally dominant as

$$|c_{22}| = |8| = 8 \le |c_{21}| + |c_{23}| = |64| + |1| = 65$$

Example 13

When are two matrices considered to be equal?

Two matrices [A] and [B] is the same (number of rows and columns are same for [A] and [B]) and $a_{ij}=b_{ij}$ for all i and j.

What would make

$$[A]=\begin{bmatrix}2 & 3\\ 6 & 7\end{bmatrix}$$

to be equal to

$$[B]=\begin{bmatrix}b_{11} & 3\\ 6 & b_{22}\end{bmatrix}$$

The two matrices [A] and [B] would be equal if $b_{11}=2$ and $b_{22}=7$.

Key Terms:

Matrix
Vector
Submatrix
Square matrix
Equal matrices
Zero matrix
Identity matrix
Diagonal matrix
Upper triangular matrix
Lower triangular matrix
Tri-diagonal matrix
Diagonally dominant matrix