Gauss-Siedel Method

Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates
Gauss-Seidel Method

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Gauss-Seidel Method

An iterative method.

Basic Procedure:
- Algebraically solve each linear equation for $x_i$
- Assume an initial guess solution array
- Solve for each $x_i$ and repeat
- Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.
The Gauss-Seidel Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to round-off error.

Also: If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.
Gauss-Seidel Method

Algorithm

A set of $n$ equations and $n$ unknowns:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
    \vdots & \ \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{align*}
\]

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for $x_1$

Second equation, solve for $x_2$

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Gauss-Seidel Method

Algorithm

Rewriting each equation

\[ x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots - a_{1n}x_n}{a_{11}} \]

\[ x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots - a_{2n}x_n}{a_{22}} \]

\[ \vdots \]

\[ x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}} \]

\[ x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}}{a_{nn}} \]

From Equation 1
From equation 2
From equation n-1
From equation n

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Gauss-Seidel Method

Algorithm

General Form of each equation

\[
x_1 = \frac{c_1 - \sum_{j=1, j\neq 1}^{n} a_{1j} x_j}{a_{11}}
\]

\[
x_2 = \frac{c_2 - \sum_{j=1, j\neq 2}^{n} a_{2j} x_j}{a_{22}}
\]

\[
x_{n-1} = \frac{c_{n-1} - \sum_{j=1, j\neq n-1}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}
\]

\[
x_n = \frac{c_n - \sum_{j=1, j\neq n}^{n} a_{nj} x_j}{a_{nn}}
\]
Gauss-Seidel Method

Algorithm

General Form for any row ‘i’

\[ c_i - \sum_{j=1}^{n} a_{ij} x_j \]

\[ x_i = \frac{c_i - \sum_{j \neq i}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n. \]

How or where can this equation be used?

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Solve for the unknowns

Assume an initial guess for \([X]\)

Use rewritten equations to solve for each value of \(x_i\).

Important: Remember to use the most recent value of \(x_i\). Which means to apply values calculated to the calculations remaining in the current iteration.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{bmatrix}
\]

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Calculate the Absolute Relative Approximate Error

\[ |\varepsilon_a|_i = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100 \]

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns.

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Gauss-Seidel Method: Example 1

The upward velocity of a rocket is given at three different times.

**Table 1** Velocity vs. Time data.

<table>
<thead>
<tr>
<th>Time, ( t ) (s)</th>
<th>Velocity ( v ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>106.8</td>
</tr>
<tr>
<td>8</td>
<td>177.2</td>
</tr>
<tr>
<td>12</td>
<td>279.2</td>
</tr>
</tbody>
</table>

The velocity data is approximated by a polynomial as:

\[
v(t) = a_1 t^2 + a_2 t + a_3 , \quad 5 \leq t \leq 12.
\]
Gauss-Seidel Method: Example 1

Using a Matrix template of the form

\[
\begin{bmatrix}
t_1^2 & t_1 & 1 \\
t_2^2 & t_2 & 1 \\
t_3^2 & t_3 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
=
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{bmatrix}
\]

The system of equations becomes

\[
\begin{bmatrix}
25 & 5 & 1 \\
64 & 8 & 1 \\
144 & 12 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
=
\begin{bmatrix}
106.8 \\
177.2 \\
279.2 \\
\end{bmatrix}
\]

Initial Guess: Assume an initial guess of

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
=
\begin{bmatrix}
1 \\
2 \\
5 \\
\end{bmatrix}
\]

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Gauss-Seidel Method: Example 1

Rewriting each equation

\[
\begin{bmatrix}
25 & 5 & 1 \\
64 & 8 & 1 \\
144 & 12 & 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} =
\begin{bmatrix}
106.8 \\
177.2 \\
279.2
\end{bmatrix}
\]

\[
a_1 = \frac{106.8 - 5a_2 - a_3}{25}
\]

\[
a_2 = \frac{177.2 - 64a_1 - a_3}{8}
\]

\[
a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}
\]
Gauss-Seidel Method: Example 1

Applying the initial guess and solving for $a_i$

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} = \begin{bmatrix}
  1 \\
  2 \\
  5
\end{bmatrix}
\]

Initial Guess

\[
a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720
\]

\[
a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510
\]

\[
a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36
\]

When solving for $a_2$, how many of the initial guess values were used?
Gauss-Seidel Method: Example 1

Finding the absolute relative approximate error

\[ |\epsilon_{a_i}| = \left| \frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}} \right| \times 100 \]

At the end of the first iteration

\[
\begin{bmatrix}
a_1 \\
\epsilon_{a_1}
\end{bmatrix} = \begin{bmatrix} 3.6720 \\
-7.8510 \\
-155.36
\end{bmatrix}
\]

The maximum absolute relative approximate error is 125.47%
Gauss-Seidel Method: Example 1

Using

\[
\begin{bmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{bmatrix} =
\begin{bmatrix}
    3.6720 \\
    -7.8510 \\
    -155.36
\end{bmatrix}
\]

from iteration #1

Iteration #2

the values of \( a_i \) are found:

\[
a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056
\]

\[
a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882
\]

\[
a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34
\]
Gauss-Seidel Method: Example 1

Finding the absolute relative approximate error

\[
|\varepsilon_a|_1 = \left| \frac{12.056 - 3.6720}{12.056} \right| \times 100 = 69.543\%
\]

\[
|\varepsilon_a|_2 = \left| \frac{-54.882 - (-7.8510)}{-54.882} \right| \times 100 = 85.695\%
\]

\[
|\varepsilon_a|_3 = \left| \frac{-798.34 - (-155.36)}{-798.34} \right| \times 100 = 80.540\%
\]

At the end of the second iteration

\[
\begin{bmatrix}
a_1 \\ a_2 \\ a_3
\end{bmatrix} = \begin{bmatrix}
12.056 \\ -54.882 \\ -798.54
\end{bmatrix}
\]

The maximum absolute relative approximate error is 85.695%
Gauss-Seidel Method: Example 1

Repeating more iterations, the following values are obtained:

| Iteration | $a_1$ | $|a_1|_1 \%$ | $a_2$ | $|a_2|_2 \%$ | $a_3$ | $|a_3|_3 \%$ |
|-----------|-------|-------------|-------|-------------|-------|-------------|
| 1         | 3.6720| 72.767      | -7.8510| 125.47      | -155.36| 103.22      |
| 2         | 12.056| 69.543      | -54.882| 85.695      | -798.34| 80.540      |
| 3         | 47.182| 74.447      | -255.51| 78.521      | -3448.9| 76.852      |
| 4         | 193.33| 75.595      | -1093.4| 76.632      | -14440| 76.116      |
| 5         | 800.53| 75.850      | -4577.2| 76.112      | -60072| 75.963      |
| 6         | 3322.6| 75.906      | -19049| 75.972      | -249580| 75.931      |

Notice – The relative errors are not decreasing at any significant rate.

Also, the solution is not converging to the true solution of

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} = \begin{bmatrix}
  0.29048 \\
  19.690 \\
  1.0857
\end{bmatrix}
\]
Gauss-Seidel Method: Pitfall

What went wrong?

Even though done correctly, the answer is not converging to the correct answer.

This example illustrates a pitfall of the Gauss-Seidel method: not all systems of equations will converge.

Is there a fix?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: \([A] \times [X] = [C]\) is diagonally dominant if:

\[
\left| a_{ii} \right| \geq \sum_{j=1, j\neq i}^{n} \left| a_{ij} \right| \quad \text{for all} \quad i \quad \text{and} \quad \left| a_{ii} \right| > \sum_{j=1, j\neq i}^{n} \left| a_{ij} \right| \quad \text{for at least one} \quad i
\]

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Gauss-Seidel Method: Pitfall

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Which coefficient matrix is diagonally dominant?

\[
\begin{bmatrix}
2 & 5.81 & 34 \\
45 & 43 & 1 \\
123 & 16 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
124 & 34 & 56 \\
23 & 53 & 5 \\
96 & 34 & 129
\end{bmatrix}
\]

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.
Gauss-Seidel Method: Example 2

Given the system of equations

\[
\begin{align*}
12x_1 + 3x_2 - 5x_3 &= 1 \\
x_1 + 5x_2 + 3x_3 &= 28 \\
3x_1 + 7x_2 + 13x_3 &= 76
\end{align*}
\]

The coefficient matrix is:

\[ [A] = \begin{bmatrix}
12 & 3 & -5 \\
1 & 5 & 3 \\
3 & 7 & 13
\end{bmatrix} \]

With an initial guess of

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\]

Will the solution converge using the Gauss-Seidel method?
Gauss-Seidel Method: Example 2

Checking if the coefficient matrix is diagonally dominant

\[
[A] = \begin{bmatrix}
12 & 3 & -5 \\
1 & 5 & 3 \\
3 & 7 & 13
\end{bmatrix}
\]

\[
|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = 3 + |-5| = 8
\]

\[
|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = 1 + |3| = 4
\]

\[
|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = 3 + |7| = 10
\]

The inequalities are all true and at least one row is strictly greater than:

Therefore: The solution should converge using the Gauss-Seidel Method
Gauss-Seidel Method: Example 2

Rewriting each equation

\[
\begin{bmatrix}
12 & 3 & -5 \\
1 & 5 & 3 \\
3 & 7 & 13 \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
28 \\
76 \\
\end{bmatrix}
\]

\[
x_1 = \frac{1-3x_2+5x_3}{12}
\]

\[
x_2 = \frac{28-x_1-3x_3}{5}
\]

\[
x_3 = \frac{76-3x_1-7x_2}{13}
\]

With an initial guess of

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
1 \\
\end{bmatrix}
\]

\[
x_1 = \frac{1-3(0)+5(1)}{12} = 0.50000
\]

\[
x_2 = \frac{28-(0.5)-3(1)}{5} = 4.9000
\]

\[
x_3 = \frac{76-3(0.50000)-7(4.9000)}{13} = 3.0923
\]
Gauss-Seidel Method: Example 2

The absolute relative approximate error

\[
|\varepsilon_a|_1 = \left| \frac{0.50000 - 1.0000}{0.50000} \right| \times 100 = 100.00\%
\]

\[
|\varepsilon_a|_2 = \left| \frac{4.9000 - 0}{4.9000} \right| \times 100 = 100.00\%
\]

\[
|\varepsilon_a|_3 = \left| \frac{3.0923 - 1.0000}{3.0923} \right| \times 100 = 67.662\%
\]

The maximum absolute relative error after the first iteration is 100%
Gauss-Seidel Method: Example 2

After Iteration #1

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} =
\begin{bmatrix}
  0.5000 \\
  4.9000 \\
  3.0923
\end{bmatrix}
\]

Substituting the x values into the equations

\[
x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679
\]

\[
x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153
\]

\[
x_3 = \frac{76 - 3(0.14679) - 7(4.9000)}{13} = 3.8118
\]

After Iteration #2

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} =
\begin{bmatrix}
  0.14679 \\
  3.7153 \\
  3.8118
\end{bmatrix}
\]

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Gauss-Seidel Method: Example 2

Iteration #2 absolute relative approximate error

\[ |\varepsilon_a|_1 = \left| \frac{0.14679 - 0.50000}{0.14679} \right| \times 100 = 240.61\% \]

\[ |\varepsilon_a|_2 = \left| \frac{3.7153 - 4.9000}{3.7153} \right| \times 100 = 31.889\% \]

\[ |\varepsilon_a|_3 = \left| \frac{3.8118 - 3.0923}{3.8118} \right| \times 100 = 18.874\% \]

The maximum absolute relative error after the first iteration is 240.61%.

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?
Gauss-Seidel Method: Example 2

Repeating more iterations, the following values are obtained:

| Iteration | $a_1$  | $|a_1|\%$ | $a_2$  | $|a_2|\%$ | $a_3$  | $|a_3|\%$ |
|-----------|--------|-----------|--------|-----------|--------|-----------|
| 1         | 0.50000| 100.00    | 4.9000 | 100.00    | 3.0923 | 67.662    |
| 2         | 0.14679| 240.61    | 3.7153 | 31.889    | 3.8118 | 18.876    |
| 3         | 0.74275| 80.236    | 3.1644 | 17.408    | 3.9708 | 4.0042    |
| 4         | 0.94675| 21.546    | 3.0281 | 4.4996    | 3.9971 | 0.65772   |
| 5         | 0.99177| 4.5391    | 3.0034 | 0.82499   | 4.0001 | 0.074383  |
| 6         | 0.99919| 0.74307   | 3.0001 | 0.10856   | 4.0001 | 0.00101   |

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$ is close to the exact solution of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.
Gauss-Seidel Method: Example 3

Given the system of equations

\[ 3x_1 + 7x_2 + 13x_3 = 76 \]
\[ x_1 + 5x_2 + 3x_3 = 28 \]
\[ 12x_1 + 3x_2 - 5x_3 = 1 \]

With an initial guess of

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

Rewriting the equations

\[ x_1 = \frac{76 - 7x_2 - 13x_3}{3} \]
\[ x_2 = \frac{28 - x_1 - 3x_3}{5} \]
\[ x_3 = \frac{1 - 12x_1 - 3x_2}{-5} \]

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Gauss-Seidel Method: Example 3

Conducting six iterations, the following values are obtained

| Iteration | $a_1$       | $|e_a_1|\%$ | $A_2$     | $|e_a_2|\%$ | $a_3$       | $|e_a_3|\%$ |
|-----------|-------------|-------------|-----------|-------------|-------------|-------------|
| 1         | 21.000      | 95.238      | 0.80000   | 100.00      | 50.680      | 98.027      |
| 2         | −196.15     | 110.71      | 14.421    | 94.453      | −462.30     | 110.96      |
| 3         | −1995.0     | 109.83      | −116.02   | 112.43      | 4718.1      | 109.80      |
| 4         | −20149      | 109.90      | 1204.6    | 109.63      | −47636      | 109.90      |
| 5         | $2.0364 \times 10^5$ | 109.89  | −12140    | 109.92      | $4.8144 \times 10^5$ | 109.89 |
| 6         | $-2.0579 \times 10^5$ | 109.89 | $1.2272 \times 10^5$ | 109.89 | $-4.8653 \times 10^6$ | 109.89 |

The values are not converging.

Does this mean that the Gauss-Seidel method cannot be used?
Gauss-Seidel Method

The Gauss-Seidel Method can still be used

The coefficient matrix is not diagonally dominant

\[
\begin{bmatrix}
3 & 7 & 13 \\
1 & 5 & 3 \\
12 & 3 & -5
\end{bmatrix}
\]

But this is the same set of equations used in example #2, which did converge.

\[
\begin{bmatrix}
12 & 3 & -5 \\
1 & 5 & 3 \\
3 & 7 & 13
\end{bmatrix}
\]

If a system of linear equations is not diagonally dominant, check to see if rearranging the equations can form a diagonally dominant matrix.

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Gauss-Seidel Method

Not every system of equations can be rearranged to have a diagonally dominant coefficient matrix.

Observe the set of equations

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 3 \\
  2x_1 + 3x_2 + 4x_3 &= 9 \\
  x_1 + 7x_2 + x_3 &= 9
\end{align*}
\]

Which equation(s) prevents this set of equation from having a diagonally dominant coefficient matrix?
Gauss-Seidel Method

Summary

- Advantages of the Gauss-Seidel Method
- Algorithm for the Gauss-Seidel Method
- Pitfalls of the Gauss-Seidel Method
Gauss-Seidel Method

Questions?
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/gauss_seidel.html
THE END

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