Saving Time Using LU Decomposition to Find Inverse

For a square matrix $[A]$ of $n \times n$ size, the computational time[1] $CT|_{DE}$ to decompose the $[A]$ matrix to $[L][U]$ form is given by

$$CT|_{DE} = T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right),$$

where

T = clock cycle time[2].

The computational time $CT|_{FS}$ to solve by forward substitution $[L][Z]=[C]$ is given by

$$CT|_{FS} = T\left(4n^2 - 4n\right)$$

The computational time $CT|_{BS}$ to solve by back substitution $[U][X]=[Z]$ is given by

$$CT|_{BS} = T\left(4n^2 + 12n\right)$$

So, the total computational time to solve a set of equations by LU decomposition is

$$\begin{aligned} CT|_{LU} &= CT|_{DE} + CT|_{FS} + CT|_{BS} \\ &= T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right) + T\left(4n^2 - 4n\right) + T\left(4n^2 + 12n\right) \\ &= T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right) \end{aligned}$$

Now let us look at the computational time taken by Gaussian elimination. The computational time $CT|_{FE}$ for the forward elimination part,

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[1] The time is calculated by first separately calculating the number of additions, subtractions, multiplications, and divisions in a procedure such as back substitution, etc. We then assume 4 clock cycles each for an add, subtract, or multiply operation, and 16 clock cycles for a divide operation as is the case for a typical AMD®-K7 chip.
http://www.isi.edu/~draper/papers/mwscas07_kwon.pdf

[2] As an example, a 1.2 GHz CPU has a clock cycle of $1/(1.2\times10^9) = 0.833333\,\text{ns}$

$$CT|_{FE} = T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right),$$

and the computational time $CT|_{BS}$ for the back substitution part is

$$CT|_{BS} = T\left(4n^2 + 12n\right)$$

So, the total computational time $CT|_{GE}$ to solve a set of equations by Gaussian Elimination is

$$CT|_{GE} = CT|_{FE} + CT|_{BS}$$

$$= T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right) + T\left(4n^2 + 12n\right)$$

$$= T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$$

The computational time for Gaussian elimination and LU decomposition is identical. This has confused me further! Why learn LU decomposition method when it takes the same computational time as Gaussian elimination, and that too when the two methods are closely related. Please convince me that LU decomposition has its place in solving linear equations!

We have the knowledge now to convince you that LU decomposition method has its place in the solution of simultaneous linear equations. Let us look at an example where the LU decomposition method is computationally more efficient than Gaussian elimination. Remember in trying to find the inverse of the matrix $[A]$ in Chapter 04.05, the problem reduces to solving n sets of equations with the n columns of the identity matrix as the RHS vector. For calculations of each column of the inverse of the $[A]$ matrix, the coefficient matrix $[A]$ matrix in the set of equation $[A][X]=[C]$ does not change. So if we use the LU decomposition method, the $[A]=[L][U]$ decomposition needs to be done only once, the forward substitution (Equation 1) n times, and the back substitution (Equation 2) n times.

So the total computational time $CT|_{inverse\,LU}$ required to find the inverse of a matrix using LU decomposition is

$$CT|_{inverse\,LU} = 1 \times CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= 1 \times T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right) + n \times T\left(4n^2 - 4n\right) + n \times T\left(4n^2 + 12n\right)$$

$$= T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$$

In comparison, if Gaussian elimination method were used to find the inverse of a matrix, the forward elimination as well as the back substitution will have to be done n times. The total computational time $CT|_{inverse\,GE}$ required to find the inverse of a matrix by using Gaussian elimination then is

$$CT|_{inverse\,GE} = n \times CT|_{FE} + n \times CT|_{BS}$$

$$= n \times T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right) + n \times T\left(4n^2 + 12n\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Clearly for large n, $CT|_{inverse\,GE} >> CT|_{inverse\,LU}$ as $CT|_{inverse\,GE}$ has the dominating terms of n^4 and $CT|_{inverse\,LU}$ has the dominating terms of n^3. For large values of n, Gaussian elimination method would take more computational time (approximately $n/4$ times – prove it) than the LU decomposition method. Typical values of the ratio of the computational time for different values of n are given in Table 1.

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\|_{inverse\,GE} / CT\|_{inverse\,LU}$	3.28	25.83	250.8	2501

Are you convinced now that LU decomposition has its place in solving systems of equations?