Prove that a unique polynomial of degree *n* or less passes through *n*+1 data points.

If the polynomial is not unique, then at least two polynomials of order n or less pass through the $n+1$ data points. Assume two polynomials of order n or less, $P_n(x)$ and $Q_n(x)$ go through (n + 1) data points, $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$

Then define

$$R_n(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ pass through all the (n + 1) data points,

$$P_n(x_i) = Q_n(x_i),\ i = 0, \ldots, n$$

Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0,\ i = 0, \ldots, n$$

The n^{th} order polynomial $R_n(x)$ has (n + 1) zeros. A polynomial of order n can have more than n zeros (in this case n+1) only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0$$

Hence

$$P_n(x) \equiv Q_n(x)$$

Extra Notes:

How can one show that if a second order polynomial has three zeros, then it is zero everywhere. If $R_2(x) = a_0 + a_1 x + a_2 x^2$, then if it has three zeros at x_1, x_2, and x_3, then

$$R_1(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = 0$$

$$R_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = 0$$

$$R_3(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = 0$$

which in matrix form gives

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above set of equations has a trivial solution, that is, $a_0 = a_1 = a_2 = 0$.

But is this the only solution?

That will be true only if the coefficient matrix is invertible.

The determinant of the coefficient matrix can be found by symbolically with forward elimination steps of Naive Gauss elimination to give.

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = x_2 x_3^2 - x_2^2 x_3 - x_1 x_3^2 + x_1^2 x_3 + x_1 x_2^2 - x_1^2 x_2$$

$$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

Since

$$x_1 \neq x_2 \neq x_3$$

the determinant is non-zero. Hence the coefficient matrix is invertible. The trivial solution $a_0 = a_1 = a_2 = 0$ is the only solution, that is, $R_2(x) \equiv 0$.